HOMEWORK

Problem 1. Let $f \in H^1(\mathbb{R}^3)$ be spherically symmetric. Prove that

$$\big||x|f(x)\big| \lesssim \|f\|_{H^1} \quad \text{for all} \quad x \in \mathbb{R}^3.$$

Use this to prove that the embedding

$$\{f \in H^1(\mathbb{R}^3) : \, f \text{ spherically symmetric}\} \hookrightarrow L^p$$

is compact for $2 < p < 6$.

Problem 2 (Stability)**.** Let I be a compact time interval and let $\tilde{v} : I \times \mathbb{R}^3 \to \mathbb{C}$ be a (strong) solution to

$$-i\tilde{v}_t + \langle\nabla\rangle\tilde{v} - \langle\nabla\rangle^{-1}(\operatorname{Re}\tilde{v})^3 + e = 0.$$

Assume that

$$\|\langle\nabla\rangle^{8/9}\tilde{v}\|_{L_t^\infty L_x^2(I\times\mathbb{R}^3)} \le M \quad \text{and} \quad \|\operatorname{Re}\tilde{v}\|_{L_t^3 L_x^6(I\times\mathbb{R}^3)} \le L$$

for some positive constants M, L. Let $t_0 \in I$ and v_0 satisfy

$$\|\langle\nabla\rangle^{8/9}[\tilde{v}(t_0) - v_0]\|_{L^2} \le M'$$

for some positive constant M'. There exists $\varepsilon_1 = \varepsilon_1(L, M, M') > 0$ such that if $\varepsilon \le \varepsilon_1$ and

$$\|e^{-i(t-t_0)\langle\nabla\rangle}[\tilde{v}(t_0) - v_0]\|_{L_t^3 L_x^6(I\times\mathbb{R}^3)} \le \varepsilon$$
$$\|\langle\nabla\rangle^{8/9} e\|_{L_t^1 L_x^2(I\times\mathbb{R}^3)} \le \varepsilon$$

then there exists a solution $v : I \times \mathbb{R}^3 \to \mathbb{C}$ to

$$-iv_t + \langle\nabla\rangle v - \langle\nabla\rangle^{-1}(\operatorname{Re} v)^3 = 0$$

with data $v(t_0) = v_0$. Furthermore, it satisfies

$$\|\tilde{v} - v\|_{L_t^3 L_x^6(I\times\mathbb{R}^3)} \le \varepsilon C(L, M, M')$$
$$\|\tilde{v} - v\|_{L_t^\infty H_x^{8/9}(I\times\mathbb{R}^3)} \le C(L, M, M').$$

Problem 3. For $\phi \in H^1(\mathbb{R}^3)$ we define

$$J(\phi) := \int_{\mathbb{R}^3} \frac{1}{2}|\nabla\phi(x)|^2 + \frac{1}{2}|\phi(x)|^2 - \frac{1}{4}|\phi(x)|^4 \, dx$$

and

$$K_2(\phi) := \int_{\mathbb{R}^3} |\nabla\phi(x)|^2 - \frac{3}{4}|\phi(x)|^4 \, dx.$$

Prove that

$$\begin{aligned} J(Q) &= \inf\{J(\phi) : \, \phi \in H^1 \setminus \{0\} \text{ and } K_2(\phi) = 0\} \\ &= \inf\{\tfrac{1}{6}\|\nabla\phi\|_2^2 + \tfrac{1}{2}\|\phi\|_2^2 : \, \phi \in H^1 \setminus \{0\} \text{ and } K_2(\phi) \le 0\}. \end{aligned}$$

Here Q denotes the unique, positive, radial solution to the elliptic equation

$$-\Delta Q + Q - Q^3 = 0.$$

Show that these infima are achieved uniquely by $\pm Q$, modulo spatial translations. Furthermore, show that

$$\{\phi \in H^1 : J(\phi) < J(Q) \text{ and } K_0(\phi) \geq 0\} = \{\phi \in H^1 : J(\phi) < J(Q) \text{ and } K_2(\phi) \geq 0\}$$

and

$$\{\phi \in H^1 : J(\phi) < J(Q) \text{ and } K_0(\phi) < 0\} = \{\phi \in H^1 : J(\phi) < J(Q) \text{ and } K_2(\phi) < 0\},$$

where

$$K_0(\phi) := \int_{\mathbb{R}^3} |\nabla\phi(x)|^2 + |\phi(x)|^2 - |\phi(x)|^4 \, dx.$$

Problem 4. Let $\phi \in H^1(\mathbb{R}^3)$ be such that $J(\phi) \leq (1-\delta)J(Q)$ for some $\delta > 0$. Prove that

1) if $K_2(\phi) \geq 0$ then $K_2(\phi) \gtrsim_\delta \|\nabla\phi\|_2^2$;

2) if $K_2(\phi) < 0$ then $K_2(\phi) \lesssim_\delta -1$.