HOMEWORK 1

Problem 1 (Sharpness of local smoothing)**.** Let $d \geq 1$. Show that

$$\sup_{f \in \mathcal{S}(\mathbb{R}^d)\setminus\{0\}} \frac{\int_{\mathbb{R}} \int_{|x|\leq 1} |e^{it\Delta} f(x)|^2 \, dx \, dt}{\|\langle \nabla \rangle^{-\frac{1}{2}-\varepsilon} f\|_2^2} = \infty$$

for any $\varepsilon > 0$.

Hint: Compute the LHS for functions of the form $f(x) = e^{-\frac{|x|^2}{4} + ix\cdot\xi_0}$.

Problem 2 (Local smoothing in 1D)**.** Let $f \in \mathcal{S}(\mathbb{R})$. Show that

$$\sup_{x \in \mathbb{R}} \| |\nabla|^{\frac{1}{2}} e^{it\partial_x^2} f \|_{L_t^2} \lesssim \|f\|_{L^2(\mathbb{R})}.$$

Problem 3 (Fraunhofer formula in L^2)**.** Let $f \in L^2(\mathbb{R}^d)$. Show that

$$\lim_{|t|\to\infty} \left\| [e^{it\Delta} f](x) - (2it)^{-\frac{d}{2}} e^{i\frac{|x|^2}{4t}} \hat{f}\big(\tfrac{x}{2t}\big) \right\|_{L_x^2} = 0.$$

Using this show that if $f \in \mathcal{S}(\mathbb{R}^d)$, then

$$\|e^{it\Delta} f\|_{L_x^p} \sim |t|^{-\frac{d}{2}+\frac{d}{p}} \quad \text{for any} \quad 2 \leq p \leq \infty,$$

where the implicit constants are allowed to depend on d, p, and f.

Problem 4 (Pointwise Fraunhofer formula)**.** Let $f \in \mathcal{S}(\mathbb{R})$. Show that

$$[e^{it\Delta} f](x) = (2it)^{-\frac{1}{2}} e^{i\frac{|x|^2}{4t}} \hat{f}\big(\tfrac{x}{2t}\big) + t^{-\frac{1}{2}-\beta} O(\|\langle x \rangle f\|_{L_x^2})$$

for any $\beta < \frac{1}{4}$.

Hint: Observe that

$$e^{it\Delta} = M(t)D(t)\mathcal{F}M(t) = M(t)D(t)\mathcal{F} + M(t)D(t)\mathcal{F}[M(t) - 1],$$

where $[M(t)f](x) = e^{i\frac{|x|^2}{4t}} f(x)$, $[D(t)f](x) = (2it)^{-\frac{1}{2}} f\big(\frac{x}{2t}\big)$, and $\mathcal{F}$ denotes the Fourier transform.