HOMEWORK 1

Problem 1. (a) Fix $2 < p < \infty$. Show that the family of functions

$$\mathcal{F} = \{f^* : f \in H^1(\mathbb{R}) \text{ with } \|f\|_2 \le 1 \text{ and } \|f'\|_2 \le 1\}$$

is precompact in $L^p(\mathbb{R})$. Here f^* denotes the spherically symmetric (=even) decreasing rearrangement of f.
(b) Show that the set

$$\mathcal{G} = \{f \in H^1(\mathbb{R}) : f \text{ is even with } \|f\|_2 \le 1 \text{ and } \|f'\|_2 \le 1\}$$

is *not* precompact in $L^p(\mathbb{R})$.

Problem 2. (Refined Fatou) Fix $d \ge 1$ and $1 \le p < \infty$. Let $\{f_n\} \subseteq L^p(\mathbb{R}^d)$ with $\limsup \|f_n\|_p < \infty$. If $f_n \to f$ almost everywhere, then

$$\int_{\mathbb{R}^d} \Big| |f_n|^p - |f_n - f|^p - |f|^p \Big| \, dx \to 0.$$

In particular, $\|f_n\|_p^p - \|f_n - f\|_p^p \to \|f\|_p^p$.

Problem 3. (Concentration compactness for Gagliardo–Nirenberg) Fix $d \ge 1$ and $2 < p < \infty$ if $d = 1, 2$ or $2 < p < \frac{2d}{d-2}$ if $d \ge 3$. Let f_n be a bounded sequence in $H^1(\mathbb{R}^d)$. Then there exist a subsequence in n, $J^* \in \{0, 1, 2, \dots\} \cup \{\infty\}$, $\{\phi^j\}_{j=1}^{J^*} \subseteq H^1$, and $\{x_n^j\}_{j=1}^{J^*} \subseteq \mathbb{R}^d$ so that along this subsequence we may write

$$f_n(x) = \sum_{j=1}^{J} \phi^j(x - x_n^j) + r_n^J(x) \quad \text{for all finite} \quad 0 \le J \le J^*,$$

satisfying the following conditions:

$$\limsup_{J \to J^*} \limsup_{n \to \infty} \|r_n^J\|_p = 0$$

$$\sup_J \limsup_{n \to \infty} \Big[\|f_n\|_{H^1}^2 - \sum_{j=1}^{J} \|\phi^j\|_{H^1}^2 - \|r_n^J\|_{H^1}^2 \Big] = 0$$

$$\sup_J \limsup_{n \to \infty} \Big[\|f_n\|_p^p - \sum_{j=1}^{J} \|\phi^j\|_p^p - \|r_n^J\|_p^p \Big] = 0$$

$$r_n^J(x + x_n^j) \to 0 \quad \text{weakly in } H^1 \text{ for all } j \le J$$

$$|x_n^j - x_n^k| \to \infty \quad \text{for each} \quad j \ne k.$$

Problem 4. For $d \ge 3$ and $f \in \dot{H}^1(\mathbb{R}^d)$,

$$\|f\|_{\frac{2d}{d-2}} \le \frac{\|W\|_{\frac{2d}{d-2}}}{\|\nabla W\|_2} \|\nabla f\|_2$$

with equality if and only if $f = \alpha W(\lambda(x - x_0))$ for some $\alpha \in \mathbb{C}$, $\lambda \in (0, \infty)$, and $x_0 \in \mathbb{R}^d$. Here W denotes

$$W(x) := \big(1 + \tfrac{1}{d(d-2)}|x|^2\big)^{-\frac{d-2}{2}},$$

which is the unique non-negative radial $\dot{H}^1$ solution to $\Delta W + W^{\frac{d+2}{d-2}} = 0$, up to scaling.