HOMEWORK 3

Problem 1 (Gagliardo–Nirenberg inequality)**.** Fix $d \geq 1$ and $0 < p < \infty$ for $d = 1, 2$ or $0 < p < \frac{4}{d-2}$ for $d \geq 3$. Show that for all $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\|f\|_{p+2}^{p+2} \leq \|f\|_2^{p+2-\frac{pd}{2}} \|\nabla f\|_2^{\frac{pd}{2}}.$$

Problem 2. Let $f \in \mathcal{S}(\mathbb{R}^d)$. Show that

$$\Big\| \frac{\partial^2 f}{\partial x_j \partial x_k} \Big\|_p \lesssim_p \|\Delta f\|_p \quad \text{for all} \quad 1 < p < \infty \quad \text{and} \quad 1 \leq j, k \leq d,$$

where $\Delta f = \sum_{j=1}^d \frac{\partial^2 f}{\partial x_j^2}$.

Problem 3. Given a Schwartz vector field $F : \mathbb{R}^3 \to \mathbb{C}^3$, define vector and scalar fields A and ϕ via

$$\hat{\phi}(\xi) = \frac{\xi \cdot \hat{F}(\xi)}{2\pi i |\xi|^2} \quad \text{and} \quad \hat{A}(\xi) = -\frac{\xi \times \hat{F}(\xi)}{2\pi i |\xi|^2}.$$

Note that ϕ and A are smooth functions, but need not be Schwartz.
(a) Show that

$$\|\phi\|_{L^q} + \|A\|_{L^q} \lesssim \|F\|_{L^p}$$

for $1 < p < q < \infty$ obeying $1 + \frac{d}{q} = \frac{d}{p}$.
(b) Show that $F = \nabla \times A + \nabla \phi$ and hence that

$$\|F\|_{L^p} \sim \|\nabla \times A\|_{L^p} + \|\nabla \phi\|_{L^p}$$

for any $1 < p < \infty$.
(c) Show that all (first-order) derivatives of all components of A are under control (not just the curl):

$$\|\partial_k A_l\|_{L^p} \lesssim \|F\|_{L^p}$$

for any $1 < p < \infty$ and any $k, l \in \{1, 2, 3\}$.

Remark. Observe that $F = \nabla \times A + \nabla \phi$ decomposes F into a divergence-free part and a curl-free part. Note however, that the choice of A is far from unique; consider $A \mapsto A + \nabla \psi$. Our choice corresponds to the Coulomb gauge: $\nabla \cdot A = 0$.

Problem 4. Let $f \in L^\infty(\mathbb{R}^d)$ and fix $0 < \alpha < 1$. Show that f is α-Hölder continuous if and only if $\|P_{\geq N} f\|_{L^\infty} \lesssim N^{-\alpha}$ for all $N \geq 1$.

Problem 5. Let $f, g \in \mathcal{S}(\mathbb{R}^d)$ and $1 < p, q, r < \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Show that

$$\Big\| \sum_{N \in 2^{\mathbb{Z}}} f_N g_{\leq N} \Big\|_{L^p} \lesssim \|f\|_{L^q} \|g\|_{L^r}.$$