PROBLEMS

Problem 1. Let us define

$$J_0(x) = \tfrac{1}{2\pi} \int_0^{2\pi} \cos\bigl(x \sin(\theta)\bigr)\, d\theta,$$

which is the Bessel function of order zero. Describe the relation of

$$f(r) \mapsto F(\rho) = 2\pi \int J_0(2\pi r\rho) f(r) r\, dr \tag{1}$$

to the action of Fourier transform on spherically-symmetric functions in two dimensions. Use this to deduce that it defines a unitary map from $L^2\bigl([0,\infty), r\, dr\bigr)$ to itself.

Problem 2. Show that the eigenvalues of the Fourier transform are contained in the set $\{1, -1, i, -i\}$.

Problem 3. We define a sequence functions on $\mathbb{R}$ by

$$\psi_n(x) = \Bigl[\frac{d}{dx} - 2\pi x\Bigr]^n e^{-\pi x^2}$$

where $n \geq 0$. Show that $\psi_n(x)$ form an orthogonal sequence of eigenfunctions for the Fourier transform on $L^2(\mathbb{R})$. In particular, show that all four potential eigenvalues listed in Problem 2 do, in fact, occur.

Problem 4. Prove Young's inequality: For $1 \leq p, q, r \leq \infty$,

$$\|f * g\|_{L^r(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)} \quad \text{whenever} \quad 1 + \tfrac{1}{r} = \tfrac{1}{p} + \tfrac{1}{q}.$$

Show that no inequalities of this type are possible for other exponents.

Problem 5. Consider the Lorentz space $L^{1,q}(\mathbb{R})$ with $1 < q < \infty$. Show that the quasinorm on $L^{1,q}(\mathbb{R})$ cannot be equivalent to any norm.

Problem 6. Let $\|\cdot\|$ denote a quasinorm on functions. Let $f_1, \dots f_N$ be functions satisfying the bounds

$$\|f_n\| \leq 2^{-\varepsilon n}$$

for some $\varepsilon > 0$. Show that

$$\Bigl\| \sum_{n=1}^{N} f_n \Bigr\| \lesssim_\varepsilon 1,$$

where the implicit constant is independent of N.
Hint: First reduce the problem to large positive ε.

Problem 7. Let $1 \leq p < \infty$ and $1 \leq q \leq \infty$ and let $f \in L^{p,q}(\mathbb{R}^n)$. We can write $f = \sum f_n$ where

$$f_n = f\chi_{\{x\colon H_{n+1} \leq |f(x)| < H_n\}} \quad \text{with} \quad H_n = \inf\{\lambda : |\{x : |f(x)| > \lambda\}| \leq 2^{n-1}\}.$$

Show that

$$\|f\|^*_{L^{p,q}} \sim \|H_n 2^{\frac{n}{p}}\|_{\ell^q(\mathbb{Z})}.$$

Hint: Show that for $H_{n+1} \leq \lambda < H_n$ we have $2^{n-1} < |\{x : |f(x)| > \lambda\}| \leq 2^n$.

Problem 8. Let $1 \le p < \infty$ and $1 \le q \le \infty$. Suppose that $H_n \ge 0$ and E_n are measurable sets with $|E_n| \le C2^n$. Show that if

$$|f| \le \sum H_n \chi_{E_n},$$

then

$$\|f\|^*_{L^{p,q}} \lesssim \|H_n 2^{\frac{n}{p}}\|_{\ell^q(\mathbb{Z})}.$$

Hint: Show that $|\{x : |f(x)| > \lambda\}| \lesssim \sup\{2^n : \sum_{m \ge n} H_m > \lambda\}$.

Problem 9. (Hölder in Lorentz spaces) Let $1 \le p, p_1, p_2 < \infty$ and $1 \le q, q_1, q_2 \le \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Show that

$$\|fg\|^*_{L^{p,q}} \lesssim \|f\|^*_{L^{p_1,q_1}} \|g\|^*_{L^{p_2,q_2}}.$$

Hint: Use the previous three problems.

Problem 10. For $1 < p, q < \infty$, we define an operator on functions on $(0, \infty)$ via

$$(Tf)(x) = |x|^{-\frac{1}{q}} \int_0^\infty f(y)|y|^{-\frac{1}{p'}}\, dy.$$

Show that T is of restricted weak type (p, q), but not of weak type (p, q).

Problem 11. Let $\omega : \mathbb{R}^d \to [0, \infty)$ given by $\omega(x) = |x|^\alpha$.
(a) Show that $\omega\, dx$ is a doubling measure if and only if $\alpha > -d$.
(b) Show that $\omega \in A_p$ with $1 < p < \infty$ if and only if $-d < \alpha < (p-1)d$.

Problem 12. Fix $1 \le p < \infty$ and let $\omega \in A_p$.
(a) Show that $M_\omega : L^1(\omega\, dx) \to L^{1,\infty}(\omega\, dx)$, where

$$M_\omega f(x) = \sup_{r>0} \frac{1}{\omega(B(x,r))} \int_{B(x,r)} |f(y)|\omega(y)\, dy.$$

(b) Show that $(Mf)^p \lesssim M_\omega(f^p)$ for all $f \ge 0$, where M denotes the Hardy-Littlewood maximal function.
(c) Conclude that $M : L^p(\omega\, dx) \to L^{p,\infty}(\omega\, dx)$.

Problem 13. The dyadic cubes in $\mathbb{R}^d$ are sets of the form

$$Q_{n,k} = [k_1 2^n, (k_1+1)2^n) \times \cdots \times [k_d 2^n, (k_d+1)2^n),$$

where n ranges over $\mathbb{Z}$ and $k \in \mathbb{Z}^d$.
(a) Given a collection of dyadic cubes with bounded maximal diameter, show that one may find a subcollection which covers the same region of $\mathbb{R}^d$, but with all cubes disjoint.
(b) Define the dyadic maximal function by

$$[M_D f](x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where the supremum is taken over all dyadic cubes that contain x. Show M_D is of weak-type $(1, 1)$ and of type (p, p) for all $1 < p \le \infty$.

Problem 14. Let M_D denote the dyadic maximal function defined above and let $Q_0 := [0, 1)^d$.
(a) For $\alpha > 0$, show that

$$|\{x \in \mathbb{R}^d : [M_D f](x) > \alpha\}| \lesssim \frac{1}{\alpha} \int_{|f| > c\alpha} |f(y)|\, dy$$

for some small constant c.
(b) Deduce that if f is supported on Q_0 and $|f|\log[2+|f|]\in L^1(Q_0)$, then $M_D f\in L^1(Q_0)$.
(c) Given $f\in L^1(Q_0)$ and $\alpha>\int_{Q_0}|f(y)|\,dy$, show that

$$|\{x\in Q_0 : [M_D f](x)>\alpha\}|\gtrsim \frac{1}{\alpha}\int_{|f|>\alpha}|f(y)|\,dy$$

Hint: perform a Calderon–Zygmund style decomposition.
(d) Deduce that if $M_D f\in L^1(Q_0)$, then $|f|\log[2+|f|]\in L^1(Q_0)$.

Problem 15 (Schur's test with weights)**.** Suppose $(X,d\mu)$ and $(Y,d\nu)$ are measure spaces and let $w(x,y)$ be a positive measurable function defined on $X\times Y$. Let $K(x,y): X\times Y\to\mathbb{C}$ satisfy

$$\sup_{x\in X}\int_Y w(x,y)^{\frac{1}{p}}|K(x,y)|\,d\nu(y)=C_0<\infty, \tag{2}$$

$$\sup_{y\in Y}\int_X w(x,y)^{-\frac{1}{p'}}|K(x,y)|\,d\mu(x)=C_1<\infty, \tag{3}$$

for some $1<p<\infty$. Then the operator defined by

$$Tf(x)=\int_Y K(x,y)f(y)\,d\nu(y)$$

is a bounded operator from $L^p(Y,d\nu)$ to $L^p(X,d\mu)$. In particular,

$$\|Tf\|_{L^p(X,d\mu)}\le C_0^{\frac{1}{p'}}C_1^{\frac{1}{p}}\|f\|_{L^p(Y,d\nu)}.$$

Remark 0.1. This is essentially a theorem of Aronszajn. When $K\ge 0$, Gagliardo has shown that the existence of a weight $w(x,y)=a(x)b(y)$ obeying (2) and (3) is necessary for the L^p boundedness of T.

Problem 16 (Hardy's inequality)**.** Let $f\in\mathcal{S}(\mathbb{R}^d)$ and $0\le s<d$. Show that

$$\Big\|\frac{f(x)}{|x|^s}\Big\|_p\lesssim \||\nabla|^s f\|_p \quad\text{for all}\quad 1<p<\frac{d}{s}.$$

Hint: Show that there exists $g\in L^p$ so that $f=|\nabla|^{-s}g$ and then use Problem 5 for the kernel $K(x,y)=|x|^{-s}|x-y|^{s-d}$.

Problem 17 (Gagliardo–Nirenberg inequality)**.** Fix $d\ge 1$ and $0<p<\infty$ for $d=1,2$ or $0<p<\frac{4}{d-2}$ for $d\ge 3$. Show that for all $f\in\mathcal{S}(\mathbb{R}^d)$,

$$\|f\|_{p+2}^{p+2}\lesssim \|f\|_2^{p+2-\frac{pd}{2}}\|\nabla f\|_2^{\frac{pd}{2}}.$$

Problem 18. Let $f\in\mathcal{S}(\mathbb{R}^d)$. Show that

$$\Big\|\frac{\partial^2 f}{\partial x_j\partial x_k}\Big\|_p\lesssim_p \|\Delta f\|_p \quad\text{for all}\quad 1<p<\infty \quad\text{and}\quad 1\le j,k\le d,$$

where $\Delta f=\sum_{j=1}^d \frac{\partial^2 f}{\partial x_j^2}$.

Problem 19. Given a Schwartz vector field $F:\mathbb{R}^3\to\mathbb{C}^3$, define vector and scalar fields A and ϕ via

$$\hat\phi(\xi)=\frac{\xi\cdot\hat F(\xi)}{2\pi i|\xi|^2}\quad\text{and}\quad \hat A(\xi)=-\frac{\xi\times\hat F(\xi)}{2\pi i|\xi|^2}.$$

Note that ϕ and A are smooth functions, but need not be Schwartz.
(a) Show that

$$\|\phi\|_{L^q} + \|A\|_{L^q} \lesssim \|F\|_{L^p}$$

for $1 < p < q < \infty$ obeying $1 + \frac{d}{q} = \frac{d}{p}$.
(b) Show that $F = \nabla \times A + \nabla\phi$ and hence that

$$\|F\|_{L^p} \sim \|\nabla \times A\|_{L^p} + \|\nabla\phi\|_{L^p}$$

for any $1 < p < \infty$.
(c) Show that all (first-order) derivatives of all components of A are under control (not just the curl):

$$\|\partial_k A_l\|_{L^p} \lesssim \|F\|_{L^p}$$

for any $1 < p < \infty$ and any $k, l \in \{1, 2, 3\}$.

Remark 0.2. Observe that $F = \nabla \times A + \nabla\phi$ decomposes F into a divergence-free part and a curl-free part. Note however, that the choice of A is far from unique; consider $A \mapsto A + \nabla\psi$. Our choice corresponds to the Coulomb gauge: $\nabla \cdot A = 0$.

Problem 20. Let $f \in L^\infty(\mathbb{R}^d)$ and fix $0 < \alpha < 1$. Show that f is α-Hölder continuous if and only if $\|P_{\geq N} f\|_{L^\infty} \lesssim N^{-\alpha}$ for all $N \geq 1$.

Problem 21. Let $f, g \in \mathcal{S}(\mathbb{R}^d)$ and $1 < p, q, r < \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Show that

$$\Big\| \sum_{N \in 2^{\mathbb{Z}}} f_N g_{\leq N} \Big\|_{L^p} \lesssim \|f\|_{L^q} \|g\|_{L^r}.$$

Problem 22 (Brezis–Wainger inequality)**.** Let $f \in \mathcal{S}(\mathbb{R}^2)$. Show that

$$\|f\|_{L^\infty} \lesssim \|f\|_{H^1} \big[1 + \log\big(\tfrac{\|f\|_{H^s}}{\|f\|_{H^1}}\big)\big]^{1/2} \quad \text{for all} \quad s > 1.$$

Recall that for $s > 0$, the Sobolev space $H^s(\mathbb{R}^2)$ is defined as the completion of $\mathcal{S}(\mathbb{R}^2)$ under the norm

$$\|f\|_{H^s} = \|\langle\nabla\rangle^s f\|_{L^2}$$

where $\langle x \rangle := (1 + |x|^2)^{1/2}$.

Problem 23. Let $d\sigma$ denote surface measure on the sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$. Show that the Fourier transform of $d\sigma$ satisfies

$$|\widehat{d\sigma}(x)| \lesssim \langle x \rangle^{-\frac{d-1}{2}}.$$

Problem 24. Prove the following dispersive estimate for the half-wave operator:

$$\big\|e^{it|\nabla|} P_N f\big\|_{L^\infty_x} \lesssim N^d (1 + |t|N)^{-\frac{d-1}{2}} \|P_N f\|_{L^1_x},$$

where the implicit constant is independent of the frequency $N \in 2^{\mathbb{Z}}$, the time $t \in \mathbb{R}$, and the initial data $f \in \mathcal{S}(\mathbb{R}^d)$.