HOMEWORK 1

Problem 1. Prove Young's inequality: For $1 \le p, q, r \le \infty$,

$$\|f \star g\|_{L^r(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)} \quad \text{whenever} \quad 1 + \tfrac{1}{r} = \tfrac{1}{p} + \tfrac{1}{q}.$$

Show that no inequalities of this type are possible for other exponents.

Problem 2. Let $\omega : \mathbb{R}^d \to [0, \infty)$ given by $\omega(x) = |x|^\alpha$.
(a) Show that $\omega\, dx$ is a doubling measure if and only if $\alpha > -d$.
(b) Show that $\omega \in A_p$ with $1 < p < \infty$ if and only if $-d < \alpha < (p-1)d$.

Problem 3. Fix $1 \le p < \infty$ and let $\omega \in A_p$.
(a) Show that $M_\omega : L^1(\omega\, dx) \to L^{1,\infty}(\omega\, dx)$, where

$$M_\omega f(x) = \sup_{r>0} \frac{1}{\omega(B(x,r))} \int_{B(x,r)} |f(y)|\omega(y)\, dy.$$

(b) Show that $(Mf)^p \lesssim M_\omega(f^p)$ for all $f \ge 0$, where M denotes the Hardy-Littlewood maximal function.
(c) Conclude that $M : L^p(\omega\, dx) \to L^{p,\infty}(\omega\, dx)$.

Problem 4. The dyadic cubes in $\mathbb{R}^d$ are sets of the form

$$Q_{n,k} = [k_1 2^n, (k_1+1)2^n) \times \cdots \times [k_d 2^n, (k_d+1)2^n),$$

where n ranges over $\mathbb{Z}$ and $k \in \mathbb{Z}^d$.
(a) Given a collection of dyadic cubes with bounded maximal diameter, show that one may find a subcollection which covers the same region of $\mathbb{R}^d$, but with all cubes disjoint.
(b) Define the dyadic maximal function by

$$[M_D f](x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where the supremum is taken over all dyadic cubes that contain x. Show M_D is of weak-type $(1,1)$ and of type (p,p) for all $1 < p \le \infty$.

Problem 5. Let M_D denote the dyadic maximal function defined above and let $Q_0 := [0,1)^d$.
(a) For $\alpha > 0$, show that

$$|\{x \in \mathbb{R}^d : [M_D f](x) > \alpha\}| \lesssim \frac{1}{\alpha} \int_{|f| > c\alpha} |f(y)|\, dy$$

for some small constant c.
(b) Deduce that if f is supported on Q_0 and $|f| \log[2 + |f|] \in L^1(Q_0)$, then $M_D f \in L^1(Q_0)$.
(c) Given $f \in L^1(Q_0)$ and $\alpha > \int_{Q_0} |f(y)|\, dy$, show that

$$|\{x \in Q_0 : [M_D f](x) > \alpha\}| \gtrsim \frac{1}{\alpha} \int_{|f| > \alpha} |f(y)|\, dy$$

Hint: perform a Calderon–Zygmund style decomposition.

(d) Deduce that if $M_D f \in L^1(Q_0)$, then $|f| \log[2 + |f|] \in L^1(Q_0)$.