HOMEWORK 1

Problem 1. Show that the eigenvalues of the Fourier transform are contained in the set $\{1, -1, i, -i\}$. *Hint:* Show that the eigenvalues are fourth roots of unity.

Problem 2. For $n \geq 0$ let $\psi_n : \mathbb{R} \to \mathbb{R}$ given by

$$\psi_n(x) = \big[-\partial_x + 2\pi x\big]^n e^{-\pi x^2}.$$

Show that $\{\psi_n\}$ is a linearly independent sequence of eigenfunctions for the Fourier transform on $L^2(\mathbb{R})$. In particular, show that all four potential eigenvalues listed in the previous problem do indeed occur.

Remark 0.1. In fact, $\{\psi_n\}$ form an orthogonal basis for $L^2(\mathbb{R})$, but this is harder to prove. They are also eigenfunctions of the Hermite operator

$$u(x) \mapsto \big[-\partial_x + 2\pi x\big]\big[\partial_x + 2\pi x\big]u(x) = -\partial_{xx}u + (4\pi^2 x^2 - 2\pi)u(x).$$

In this setting the operator $-\partial_x + 2\pi x$ is known as the creation operator, while its formal adjoint $\partial_x + 2\pi x$ is known as the annihilation operator.

Problem 3. Prove Young's inequality: For $1 \leq p, q, r \leq \infty$,

$$\|f \star g\|_{L^r(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)} \quad \text{whenever} \quad 1 + \tfrac{1}{r} = \tfrac{1}{p} + \tfrac{1}{q}.$$

Show that no inequalities of this type are possible for other exponents.

Problem 4. The dyadic cubes in $\mathbb{R}^d$ are sets of the form

$$Q_{n,k} = [k_1 2^n, (k_1 + 1)2^n) \times \cdots \times [k_d 2^n, (k_d + 1)2^n),$$

where n ranges over $\mathbb{Z}$ and $k \in \mathbb{Z}^d$.
(a) Given a collection of dyadic cubes with bounded maximal diameter, show that one may find a subcollection which covers the same region of $\mathbb{R}^d$, but with all cubes disjoint.
(b) Define the dyadic maximal function by

$$[M_D f](x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where the supremum is taken over all dyadic cubes that contain x. Show M_D is of weak-type $(1, 1)$ and of type (p, p) for all $1 < p \leq \infty$.

Problem 5. Let M_D denote the dyadic maximal function defined above and let $Q_0 := [0, 1)^d$.
(a) For $\alpha > 0$, show that

$$|\{x \in \mathbb{R}^d : [M_D f](x) > \alpha\}| \lesssim \frac{1}{\alpha} \int_{|f| > c\alpha} |f(y)|\, dy$$

for some small constant c.
(b) Deduce that if f is supported on Q_0 and $|f| \log[2 + |f|] \in L^1(Q_0)$, then $M_D f \in L^1(Q_0)$.

(c) Given $f \in L^1(Q_0)$ and $\alpha > \int_{Q_0} |f(y)|\, dy$, show that

$$|\{x \in Q_0 : [M_D f](x) > \alpha\}| \gtrsim \frac{1}{\alpha} \int_{|f|>\alpha} |f(y)|\, dy$$

Hint: perform a Calderon–Zygmund stype decomposition.
(d) Deduce that if $M_D f \in L^1(Q_0)$, then $|f| \log[2 + |f|] \in L^1(Q_0)$.

Problem 6. Given a non-negative $f \in L^1(\mathbb{R})$, define

$$[M_R f](x) = \sup_{t>0} \frac{1}{t} \int_0^t f(x+s)\, ds.$$

Fix $\alpha > 0$ and let $S = \{x \in \mathbb{R} : [M_R f](x) > \alpha\}$. Show that $|S| = \alpha^{-1} \int_S f(y)\, dy$.
Hint: Show that S is open.