HOMEWORK 8

Due on Wednesday, May 27th, in class.

Exercise 1. (10 points) Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a, b] \to [a, b]$ be continuous. Show that there exists $x_0 \in [a, b]$ such that $f(x_0) = x_0$.

Exercise 2. (20 points) Given a metric space (X, d_X), let $C(X)$ denote the set of continuous bounded functions $f : X \to \mathbb{R}$. For $f, g \in C(X)$, we define

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

a) Prove that $(C(X), d)$ is a metric space.
b) Show that $C(X)$ is complete, connected, but not compact.

Exercise 3. (10 points) Consider the subset of $C([0, 1])$ defined as follows:

$$X = \{f : [0, 1] \to \mathbb{R} |\, f(0) = 0, |f(x) - f(y)| \leq |x - y|\}.$$

Prove that X is compact.

Exercise 4. (10 points) For $n \geq 1$ let

$$f_n : [0, \infty) \to \mathbb{R}, \quad f_n(x) = \frac{nx^2 + 1}{nx + 1}.$$

Study the pointwise and uniform convergence of f_n on each of the intervals $[0, \infty)$, $(0, \infty)$, $[1, \infty)$.

Exercise 5. (10 points) For $n \geq 1$ let $f_n : [1, 2] \to \mathbb{R}$ be defined as follows: for any $x \in [1, 2]$,

$$f_1(x) = 0, \quad f_{n+1}(x) = \sqrt{x + f_n(x)}, \quad \forall n \geq 1.$$

Prove that $\{f_n\}_{n \geq 1}$ converges uniformly to $f(x) = \frac{1+\sqrt{1+4x}}{2}$.

Exercise 6. (20 points) Let $f, g : [-1, 1] \to [-1, 1]$ be defined as follows:

$$f(x) = \begin{cases} \frac{x-1}{2}, & x \in [-1, 0] \\ x - \frac{1}{2} \sin\left(\frac{\pi}{x}\right), & x \in (0, \frac{1}{2}] \\ x, & x \in [\frac{1}{2}, 1] \end{cases}$$

and

$$g(x) = \begin{cases} \frac{1-x}{2}, & x \in [-1, 0] \\ -x - \frac{1}{2} \sin\left(\frac{\pi}{x}\right), & x \in (0, \frac{1}{2}] \\ -x, & x \in [\frac{1}{2}, 1]. \end{cases}$$

Let $A = \{(x, f(x)) |\, x \in [-1, 1]\}$ be the graph of f and let $B = \{(x, g(x)) |\, x \in [-1, 1]\}$ be the graph of g. Prove that A, B are disjoint connected subsets of $[-1, 1] \times [-1, 1]$ such that $\{(-1, -1), (1, 1)\} \subseteq A$ and $\{(-1, 1), (1, -1)\} \subseteq B$.

Exercise 7. (10 points) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{q}, & \text{if} \quad x = \frac{p}{q} \text{ with } (p, q) = 1. \end{cases}$$

Prove that f is continuous on $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous on $\mathbb{Q}$.

Exercise 8. (10 points) Let (X, d) be a metric space and let $f, g : X \to \mathbb{R}$ be two continuous functions.
a) Prove that the set $\{x \in X | \, f(x) < g(x)\}$ is open.
b) Prove that if the set $\{x \in X | \, f(x) \leq g(x)\}$ is dense in X, then $f(x) \leq g(x)$ for all $x \in X$.