HOMEWORK 7

Due on Monday, May 18th, in class.

Exercise 1. (10 points) Let $f : [0, \infty) \to [0, \infty)$ be continuous with $f(0) = 0$. Show that if

$$f(t) \leq 1 + \tfrac{1}{10} f(t)^2 \qquad \text{for all } t \in [0, \infty),$$

then f is uniformly bounded throughout $[0, \infty)$.

Exercise 2. (10 points) Let (X, d) be a metric space and define a binary relation on X via

$$x \sim y \qquad \Longleftrightarrow \qquad \text{there is a connected set } B \subseteq X \text{ with } x, y \in B.$$

(a) Show that this is an equivalence relation. The equivalence classes are known as the *connected components* of X.
(b) Show that every connected component is closed.

Exercise 3. (10 points) Suppose A and B are compact subsets of a metric space. Show that $A \cap B$ and $A \cup B$ are also compact.

Exercise 4. (10 points) Let $\{A_i\}_{i \in I}$ be an infinite family of closed sets with the finite intersection property. Assuming that one member of this family is compact, show that $\cap_{i \in I} A_i \neq \varnothing$.

Exercise 5. (10 points) Let (X, d) be a metric space and let $A \subseteq X$ be a compact subset. Show that
(a) For any $y \in X$ there exists $x \in A$ so that $d(y, A) = d(y, x)$.
(b) If $B \subseteq X$ and $d(A, B) = 0$ then $A \cap \bar{B} \neq \varnothing$.

Exercise 6. (20 points) Let (X, d_X) be a compact metric space.
(a) Verify that

$$d_Y(f, g) = \sum_{n \in \mathbb{Z}} 2^{-|n|} d_X\big(f(n), g(n)\big)$$

defines a metric on $Y = \{f : \mathbb{Z} \to X\}$.
(b) Show that Y is compact.

Exercise 7. (10 points) Show that the function

$$H(x, y) = x^2 + y^2 + |x - y|^{-1}$$

achieves its global minimum somewhere on the set $\{(x, y) \in \mathbb{R}^2 | x \neq y\}$.

Exercise 8. (10 points) Define $A \subseteq \ell^2$ by

$$A = \big\{x \in \ell^2 | \textstyle\sum_{n=1}^{\infty} n|x_n|^2 \leq 1\big\}.$$

Show that A is compact.

Exercise 9. (10 points) Let A be a subset of a complete metric space. Assume that for all $\varepsilon > 0$, there exists a compact set A_ε so that

$$\forall x \in A, \quad d(x, A_\varepsilon) < \varepsilon.$$

Show that $\bar{A}$ is compact.