HOMEWORK 1

Due on Monday, April 6th, in class.

Exercise 1. Let $v, w \in \mathbb{E}^n$. Prove the parallelogram law
$$\|v+w\|^2 + \|v-w\|^2 = 2\big(\|v\|^2 + \|w\|^2\big).$$

Exercise 2. Let $v, w \in \mathbb{E}^n$. Prove that
$$\|v+w\|^2 - \|v-w\|^2 = 4\langle v, w\rangle,$$
and hence $\langle v, w\rangle = 0$ if and only if $\|v+w\| = \|v-w\|$. Geometrically, this means that the diagonals of a parallelogram are of equal length if and only if the parallelogram is a rectangle.

Exercise 3. Let $\{v_1, \dots, v_k\} \subseteq \mathbb{E}^n \backslash \{0\}$ be pairwise orthogonal vectors. Prove that $\{v_1, \dots, v_k\}$ is a linearly independent set.

Exercise 4. Let $v, w \in \mathbb{E}^3$ be linearly independent vectors. Show
$$\langle v, v \times w\rangle = \langle w, v \times w\rangle = 0$$
$$\|v \times w\| = \|v\|\,\|w\|\,\sin\theta,$$
where θ is the angle formed by v and w.

Exercise 5. Prove that three distinct points $p_1, p_2, p_3 \in \mathbb{R}^3$ lie on a line if and only if $(p_2 - p_1) \times (p_3 - p_1) = 0$. Consequently, the line through two distinct points $p_1, p_2 \in \mathbb{R}^3$ consists of all points $p \in \mathbb{R}^3$ such that $(p_1 - p) \times (p_2 - p) = 0$.

Exercise 6. Let $v_1 = (2, -1, 1)$, $v_2 = (1, 2, -1)$, and $v_3 = (1, 1, -2)$ be vectors in $\mathbb{E}^3$. Find all vectors $v \in \mathbb{E}^3$ of the form $\alpha v_2 + \beta v_3$ for $\alpha, \beta \in \mathbb{R}$, which are orthogonal to v_1 and have length 1.

Exercise 7. A vector $v \in \mathbb{E}^n$ has length 6. A vector $w \in \mathbb{E}^n$ has the property that for every pair of scalars $\alpha, \beta \in \mathbb{R}$, the vectors $\alpha v + \beta w$ and $4\beta v - 9\alpha w$ are orthogonal. Compute the length of w and the length of $2v + 3w$.

Exercise 8. Consider the vector space $C([-1, 1])$ of continuous functions $f : [-1, 1] \to \mathbb{R}$. Check that
$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$$
defines an inner product on this space.

Exercise 9. Using the inner product defined in Exercise 8, find an orthonormal basis for the set of polynomials $\{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}$.

Exercise 10. Using the norm induced by the inner product defined in Exercise 8, find the polynomial of degree at most two which is closest to $\sin(\pi x)$.