HOMEWORK 5

Due on Wednesday, February 11th, in class.

Exercise 1. Let $q \in \mathbb{Z}$ such that $q \geq 2$. Recall the equivalence relation on $\mathbb{Z}$ defined as follows: for $m, n \in \mathbb{Z}$, we write $m \sim n$ if $q|(m-n)$. For $n \in \mathbb{Z}$, denote by $C(n)$ the equivalence class of n. Let $\mathbb{Z}/q\mathbb{Z}$ denote the set of equivalence classes. Define addition and multiplication on $\mathbb{Z}/q\mathbb{Z}$ as follows:

$$C(n) + C(m) = C(n+m) \qquad \text{and} \qquad C(n) \cdot C(m) = C(nm).$$

1) Prove that addition and multiplication are well defined, that is, the result is independent of the representatives chosen from the equivalence classes.
2) Verify that with these operations $\mathbb{Z}/q\mathbb{Z}$ is a commutative ring with 1.
3) Show that if q is a prime number then $\mathbb{Z}/q\mathbb{Z}$ is a field.
4) Show that there is no order relation on $\mathbb{Z}/q\mathbb{Z}$ that makes it an ordered field.

Exercise 2. Define two internal laws of composition on $R = \mathbb{R} \times \mathbb{R}$ as follows

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$
$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1).$$

1) Show that with these operations R is a commutative field.
2) Show that there is no order relation on R that makes R an ordered ring.

Exercise 3. (In this exercise we will see how for any integral domain R there is a natural (smallest) field F containing it, called the field of fractions of R.) Let R be an integral domain and let $\sim$ be the equivalence relation on the set $A = \{(a, b) \in R \times R | \, b \neq 0\}$ defined as follows: $(a, b) \sim (c, d)$ if $ad = bc$. For any $(a, b) \in A$ let $[(a, b)]$ denote its equivalence class and let F denote the set of equivalence classes. Define two internal laws of composition on F as follows:

$$[(a, b)] + [(c, d)] = [(ad + bc, bd)]$$
$$[(a, b)] \cdot [(c, d)] = [(ac, bd)].$$

1) Prove that $\sim$ is an equivalence relation on A.
2) Show that the internal laws of composition defined above are well defined.
3) Show that with these internal laws of composition, F is a commutative field.

Exercise 4. Solve 3.1.9 from the textbook.

Exercise 5. Solve 3.1.14 from the textbook.