HOMEWORK 9

Exercise 1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$.
(a) Show that if P is a partition of $[a, b]$, then

$$U(f^2, P) - L(f^2, P) \leq 2M\big[U(f, P) - L(f, P)\big].$$

(b) Deduce that if f is integrable on $[a, b]$, then f^2 is also integrable on $[a, b]$.

(c) Prove that if f and g are two integrable functions on $[a, b]$, then the product fg is integrable on $[a, b]$.

Exercise 2. Let $f, g : [a, b] \to \mathbb{R}$ be two Riemann integrable functions such that the set $\{x \in [a, b] : \ f(x) = g(x)\}$ is dense in $[a, b]$. Show that

$$\int_a^b f(x)\, dx = \int_a^b g(x)\, dx.$$

Exercise 3. Suppose $f : [1, \infty) \to \mathbb{R}$ is Riemann integrable on $[1, a]$ for all $a > 1$. If

$$\lim_{a\to\infty} \int_1^a f(x)\, dx$$

exists and is finite, we say the integral $\int_1^\infty f(x)\, dx$ converges and we write

$$\int_1^\infty f(x)\, dx = \lim_{a\to\infty} \int_1^a f(x)\, dx.$$

Now assume $f : [1, \infty) \to \mathbb{R}$ is non-negative and decreasing. Show that

$$\int_1^\infty f(x)\, dx \text{ converges if and only if } \sum_{n\geq 1} f(n) \text{ converges.}$$

Exercise 4. Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function such that $f \geq 0$ and

$$\int_a^b f(x)\, dx = 0.$$

Show that if $x \in [a, b]$ is a point of continuity for f then $f(x) = 0$.

Exercise 5. Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function such that

$$\int_a^b x^n f(x)\, dx = 0 \quad \text{for all} \quad n \geq 0.$$

Show that if $x \in [a, b]$ is a point of continuity for f then $f(x) = 0$.

Exercise 6. Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable functions such that g is monotone. Show that there exists $x_0 \in [a, b]$ such that

$$\int_a^b f(x)g(x)\, dx = g(a) \int_a^{x_0} f(x)\, dx + g(b) \int_{x_0}^b f(x)\, dx.$$

Hint: Show that if g is monotonically decreasing on $[a, b]$ with $g(b) = 0$, then

$$g(a) \inf_{x \in [a,b]} \int_a^x f(t)\, dt \leq \int_a^b f(x)g(x)\, dx \leq g(a) \sup_{x \in [a,b]} \int_a^x f(t)\, dt.$$

Exercise 7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and define $F : \mathbb{R} \to \mathbb{R}$ via

$$F(x) = \int_{x-1}^{x+1} f(t)\, dt.$$

Show that F is differentiable and compute its derivative.

Exercise 8. For $n \geq 1$, let $f_n : [0, 1] \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} n, & \text{if} \quad 0 \leq x \leq \frac{1}{n} \\ 0, & \text{if} \quad \frac{1}{n} < x \leq 1. \end{cases}$$

(a) Show that

$$\lim_{n \to \infty} f_n(x) = 0 \quad \text{for all} \quad x \in (0, 1].$$

(b) Show that for each $n \geq 1$, f_n is Riemann integrable and satisfies

$$\int_0^1 f_n(x)\, dx = 1.$$