HOMEWORK 8

Exercise 1. Assume $f : (a, b) \to \mathbb{R}$ is a twice differentiable function. Show that for any $x \in (a, b)$, the limit

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}$$

exists and equals $f''(x)$.

Exercise 2. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and three times differentiable on (a, b). If

$$f(a) = f'(a) = f(b) = f'(b) = 0,$$

then there exists $c \in (a, b)$ such that $f'''(c) = 0$.

Exercise 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that

$$f(x) \geq 0 \quad \text{and} \quad f''(x) \leq 0 \quad \text{for all} \quad x \in \mathbb{R}.$$

Show that f is constant.

Exercise 4. We say that a function $f : [a, b] \to \mathbb{R}$ is a convex function if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad \text{for all } x, y \in [a, b] \text{ and for all } t \in [0, 1].$$

Show that for any $x \in (a, b)$ the one-sided limits

$$\lim_{y \searrow x} \frac{f(y) - f(x)}{y - x} \quad \text{and} \quad \lim_{y \nearrow x} \frac{f(y) - f(x)}{y - x}$$

exist and are finite.

Hint: Show that for all $a \leq x < y < z \leq b$ we have

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(x)}{z - x} \leq \frac{f(z) - f(y)}{z - y}.$$

Exercise 5. Let $f : [a, b] \to \mathbb{R}$ be a function such that

$$L(x) = \lim_{y \to x} f(x)$$

is well defined and finite for all $x \in [a, b]$ (with one-sided limits at $x = a, b$).

(a) Show that L is continuous on $[a, b]$.

(b) Show that the set $\{x \in [a, b] : f(x) \neq L(x)\}$ is at most countable.

Exercise 6. Let (X, d) be a complete metric space and let $f : X \to X$ be a function. Writing f^n for the n-th iterate of f, let

$$c_n = \sup_{x \neq y} \frac{d(f^n(x), f^n(y))}{d(x, y)}.$$

Assume $\sum_{n \geq 1} c_n < \infty$. Show that f has a fixed point in X and that this fixed point is unique.

Exercise 7. Let $f_n : [-1,1] \to [0,1]$ be continuous functions. Assume that for every $x \in [-1,1]$, the sequence $\{f_n(x)\}_{n\geq 1}$ is decreasing and

$$\lim_{n\to\infty} f_n(x) = 0.$$

For $n \geq 1$ and $x \in [-1,1]$, let

$$g_n(x) = \sum_{m=1}^{n} (-1)^m f_m(x).$$

(a) Show that $\{g_n(x)\}_{n\geq 1}$ converges to some $g(x) \in \mathbb{R}$ for all $x \in [-1,1]$.

(b) Show that the function g is continuous on $[-1,1]$.