HOMEWORK 7

Exercise 1. Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Show that for any $c \in (a, b)$ that is not a point of maximum or minimum for f' there exist $x_1, x_2 \in (a, b)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Exercise 2. Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Assume that f' is strictly increasing. Show that for any $c \in (a, b)$ such that $f'(c) = 0$ there exist $x_1, x_2 \in [a, b]$, $x_1 < c < x_2$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Exercise 3. Assume $f : (1, \infty) \to \mathbb{R}$ is differentiable. If

$$\lim_{x\to\infty} f(x) = 1 \quad \text{and} \quad \lim_{x\to\infty} f'(x) = c,$$

prove that $c = 0$.

Exercise 4. Let $f : (0, 1) \to \mathbb{R}$ be a differentiable function such that

$$|f'(x)| < 1 \quad \text{for all} \quad x \in (0, 1).$$

For $n \geq 2$ let $a_n = f(\frac{1}{n})$. Show that

$$\lim_{n\to\infty} a_n \text{ exists.}$$

Exercise 5. Let $f : (a, b) \to \mathbb{R}$ be differentiable and let $c \in (a, b)$. Suppose that $\lim_{x\to c} f'(x)$ exists and is finite. Show this limit must be $f'(c)$.

Exercise 6. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function, that is,

$$\sup\Big\{ \tfrac{|f(x)-f(y)|}{|x-y|} : x, y \in \mathbb{R} \text{ with } x \neq y \Big\} < \infty.$$

Suppose that for every $x \in \mathbb{R}$,

$$\lim_{n\to\infty} n\big[f\big(x + \tfrac{1}{n}\big) - f(x)\big] = \lim_{n\to\infty} n\big[f\big(x - \tfrac{1}{n}\big) - f(x)\big] = 0.$$

Prove that f is differentiable on $\mathbb{R}$.

Exercise 7. Let (X, d) be a metric space. Prove that the following three statements are equivalent:

(a) X is separable, that is, X admits a countable dense set.

(b) X is second countable, that is, there exists a countable basis for the topology. Note that a collection $\mathcal{G}$ of open sets is a basis for the topology if every open set in X can be written as union of elements in $\mathcal{G}$.

(c) X has the Lindelöf covering property, that is, every open cover of X has a countable subcover.

Exercise 8. Prove that if (X, d) is a separable metric space, then so is (Y, d) for every subset $Y \subseteq X$.