HOMEWORK 6

Exercise 1. Let (X, d) be a metric space with at least two points and let $\mathcal{A} \subseteq C(X)$ be an algebra that is dense in the metric space $C(X)$.

(a) Show that $\mathcal{A}$ separates points on X.

(b) Show that $\mathcal{A}$ vanishes at no point in X.

Exercise 2. (a) Show that given any continuous function $f : [0,1] \times [0,1] \to \mathbb{R}$ and any $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and functions $g_1, \dots g_n, h_1, \dots, h_n \in C([0,1])$ such that

$$\left| f(x,y) - \sum_{k=1}^{n} g_k(x) h_k(y) \right| < \varepsilon \quad \text{for all} \quad (x,y) \in [0,1] \times [0,1].$$

(b) If $f(x,y) = f(y,x)$ for all $(x,y) \in [0,1] \times [0,1]$, can this be done with $g_k = h_k$ for each $1 \leq k \leq n$? Justify your answer!

Exercise 3. Show that given any continuous function $f : [0,1] \to \mathbb{R}$ and any $\varepsilon > 0$, there exist $n \in \mathbb{N}$, coefficients $c_1, \dots, c_n \in \mathbb{R}$, and $\lambda_1, \dots \lambda_n \in (0, \infty)$ such that

$$\left| f(x) - \sum_{k=1}^{n} c_k e^{-\lambda_k |x|^2} \right| < \varepsilon \quad \text{for all} \quad x \in [0,1].$$

Exercise 4. We define the following polynomials: $P_0 = 0$ and

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - [P_n(x)]^2}{2} \quad \text{for all } n \geq 0.$$

(a) Show that $0 \leq P_n(x) \leq P_{n+1}(x) \leq |x|$ for all $|x| \leq 1$ and all $n \geq 0$.

(b) Show that

$$\lim_{n \to \infty} P_n(x) = |x| \quad \textit{uniformly} \text{ for } |x| \leq 1.$$

Exercise 5. Assume $f : [a,b] \to \mathbb{R}$ is a continuous function on the closed interval $[a,b]$ and differentiable on the open interval (a,b) with $f(a) = f(b) = 0$. Prove that for every $\lambda \in \mathbb{R}$ there exists $x_0 \in (a,b)$ such that $f'(x_0) = \lambda f(x_0)$.

Exercise 6. Let $f : [0,1] \to \mathbb{R}$ be a continuous function on the closed interval $[0,1]$ and differentiable on the open interval $(0,1)$. Assume that $f(0) = 0$ and f' is an increasing function on $(0,1)$. Show that

$$g(x) = \frac{f(x)}{x}$$

is an increasing function on $(0,1)$.