HOMEWORK 5

Exercise 1. Let $\{f_n\}_{n\geq 1} \subset C([0,1])$. Show that the following statements are equivalent:

(a) $\{f_n\}_{n\geq 1}$ converges uniformly on $[0,1]$,

(b) $\{f_n\}_{n\geq 1}$ is equicontinuous on $[0,1]$ and converges pointwise on $[0,1]$.

Exercise 2. For $n \geq 1$, let $f_n : [0,1] \to \mathbb{R}$ be given by

$$f_n(x) = \frac{\sin(nx)}{\sqrt{n}}.$$

Show that $\{f_n\}_{n\geq 1}$ is equicontinuous on $[0,1]$.

Exercise 3. Prove that a polynomial of degree n is uniformly continuous on $\mathbb{R}$ if and only if $n = 0$ or $n = 1$.

Exercise 4. Let $f : [0,1] \to [0,1]$ be a continuous function such that $f(0) = 0$ and $f(1) = 1$. Consider the sequence of functions $f_n : [0,1] \to [0,1]$ defined as follows:

$$f_1 = f \quad \text{and} \quad f_{n+1} = f \circ f_n \text{ for } n \geq 1.$$

Prove that if $\{f_n\}_{n\geq 1}$ converges uniformly, then $f(x) = x$ for all $x \in [0,1]$.

Exercise 5. Let

$$\mathcal{F} = \big\{ f \in C(\mathbb{R}) : \lim_{|x|\to\infty} f(x) = 0 \big\}.$$

Show that $\mathcal{F}$ is closed in $C(\mathbb{R})$.

Exercise 6. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-x^2}$. Find

(a) an open set $D \subseteq \mathbb{R}$ such that $f(D)$ is not open;

(b) a closed set $F \subseteq \mathbb{R}$ such that $f(F)$ is not closed;

(c) a set $A \subseteq \mathbb{R}$ such that $f(\bar{A}) \neq \overline{f(A)}$.

Exercise 7. Let $\{F_n\}_{n\geq 1}$ be a sequence of closed sets such that $F_n \subseteq F_{n+1}$ for all $n \geq 1$. Set $F = \cup_{n\geq 1} F_n$ and $F_0 = \emptyset$. For $n \geq 1$ we define

$$A_n = \big[(F_n \setminus F_{n-1}) \setminus \mathrm{Int}(F_n \setminus F_{n-1})\big] \cup \big[\mathrm{Int}(F_n \setminus F_{n-1}) \cap \mathbb{Q}\big].$$

Let $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 2^{-n} & \text{if } x \in A_n \\ 0 & \text{if } x \notin \cup_{n\geq 1} A_n. \end{cases}$$

Show that f is discontinuous on F and continuous on $\mathbb{R} \setminus F$.

Remark: This exercise shows that given any F_σ subset of $\mathbb{R}$, there is a function whose set of discontinuities is precisely that set.