Exercise 1. Let \( \{f_n\}_{n \geq 1} \subset C([0,1]) \). Show that the following statements are equivalent:

(a) \( \{f_n\}_{n \geq 1} \) converges uniformly on \([0,1]\),
(b) \( \{f_n\}_{n \geq 1} \) is equicontinuous on \([0,1]\) and converges pointwise on \([0,1]\).

Exercise 2. For \( n \geq 1 \), let \( f_n : [0,1] \to \mathbb{R} \) be given by

\[
f_n(x) = \frac{\sin(nx)}{\sqrt{n}}.
\]

Show that \( \{f_n\}_{n \geq 1} \) is equicontinuous on \([0,1]\).

Exercise 3. Prove that a polynomial of degree \( n \) is uniformly continuous on \( \mathbb{R} \) if and only if \( n = 0 \) or \( n = 1 \).

Exercise 4. Let \( f : [0,1] \to [0,1] \) be a continuous function such that \( f(0) = 0 \) and \( f(1) = 1 \). Consider the sequence of functions \( f_n : [0,1] \to [0,1] \) defined as follows:

\[
f_1 = f \quad \text{and} \quad f_{n+1} = f \circ f_n \quad \text{for} \quad n \geq 1.
\]

Prove that if \( \{f_n\}_{n \geq 1} \) converges uniformly, then \( f(x) = x \) for all \( x \in [0,1] \).

Exercise 5. Let

\[
\mathcal{F} = \{ f \in C(\mathbb{R}) : \lim_{|x| \to \infty} f(x) = 0 \}.
\]

Show that \( \mathcal{F} \) is closed in \( C(\mathbb{R}) \).

Exercise 6. Let \( f : \mathbb{R} \to \mathbb{R}, f(x) = e^{-x^2} \). Find

(a) an open set \( D \subseteq \mathbb{R} \) such that \( f(D) \) is not open;
(b) a closed set \( F \subseteq \mathbb{R} \) such that \( f(F) \) is not closed;
(c) a set \( A \subseteq \mathbb{R} \) such that \( f(A) \neq f(A) \).

Exercise 7. Let \( \{F_n\}_{n \geq 1} \) be a sequence of closed sets such that \( F_n \subseteq F_{n+1} \) for all \( n \geq 1 \). Set \( F = \bigcup_{n \geq 1} F_n \) and \( F_0 = \emptyset \). For \( n \geq 1 \) we define

\[
A_n = [(F_n \setminus F_{n-1}) \setminus \text{Int}(F_n \setminus F_{n-1})] \cup \text{Int}(F_n \setminus F_{n-1}) \cap \mathbb{Q}.
\]

Let \( f : \mathbb{R} \to \mathbb{R} \) given by

\[
f(x) = \begin{cases} 2^{-n} & \text{if } x \in A_n \\ 0 & \text{if } x \notin \bigcup_{n \geq 1} A_n. \end{cases}
\]

Show that \( f \) is discontinuous on \( F \) and continuous on \( \mathbb{R} \setminus F \).

Remark: This exercise shows that given any \( F_\alpha \) subset of \( \mathbb{R} \), there is a function whose set of discontinuities is precisely that set.