HOMEWORK 4

Exercise 1. For $n \geq 1$ let $f_n : [1,2] \to \mathbb{R}$ be defined as follows: for any $x \in [1,2]$,

$$f_1(x) = 0, \quad f_{n+1}(x) = \sqrt{x + f_n(x)}, \quad \forall n \geq 1.$$

Prove that $\{f_n\}_{n\geq 1}$ converges uniformly to $f(x) = \frac{1+\sqrt{1+4x}}{2}$.

Exercise 2. For $n \geq 1$ let

$$f_n : [0,\infty) \to \mathbb{R}, \quad f_n(x) = \frac{nx^2+1}{nx+1}.$$

Study the pointwise and uniform convergence of f_n on each of the intervals $[0,\infty)$, $(0,\infty)$, $[1,\infty)$.

Exercise 3. Given a metric space (X,d), let $C(X)$ denote the set of bounded and continuous functions $f : X \to \mathbb{R}$. For $f, g \in C(X)$, we define

$$d(f,g) = \sup_{x\in X} |f(x) - g(x)|.$$

a) Prove that $(C(X), d)$ is a metric space.

b) Show that $C(X)$ is complete, connected, but not compact.

Exercise 4. Let $f : [0,1] \to \mathbb{R}$. We say that f is Hölder continuous of order $\alpha \in (0,1)$ and write $f \in C^\alpha([0,1])$ if

$$\|f\|_{C^\alpha} := \sup\{|f(x)| : x \in [0,1]\} + \sup\left\{\tfrac{|f(x)-f(y)|}{|x-y|^\alpha} : x, y \in [0,1] \text{ with } x \neq y\right\} < \infty.$$

For $f, g \in C^\alpha([0,1])$, we define

$$d(f,g) = \|f - g\|_{C^\alpha}.$$

(a) Show that $(C^\alpha([0,1]), d)$ is a complete metric space.

(b) Prove that any bounded sequence in $C^{1/2}([0,1])$ admits a subsequence that converges in $C^{1/3}([0,1])$.

Exercise 5. Let X and Y be two metric spaces and let $f : X \to Y$ be a function. Assume A and B are open subsets of X such that $f|_A$ is continuous and $f|_B$ is continuous.

(a) Show that $f|_{A\cup B}$ is continuous.

(b) Is this result still true if A and B were both closed subsets of X?

(c) Is the result true for the union of an infinite number of open sets?

(d) Is the result true for the union of an infinite number of closed sets?

Exercise 6. Let $f : [0,1] \to \mathbb{R}$ be a function with Darboux's property such that for any $y \in \mathbb{R}$, the set $f^{-1}(\{y\})$ is closed. Prove that f is continuous.

Exercise 7. Let $f, g : [a,b] \to [a,b]$ be two continuous functions such that $f \circ g = g \circ f$. Show that there exists $x_0 \in [a,b]$ such that $f(x_0) = g(x_0)$.