Exercise 1. Let \( f : [0, \infty) \to [0, \infty) \) be a continuous function with \( f(0) = 0 \). Show that if
\[
f(t) \leq 1 + \frac{1}{10}f(t)^3
\]
for all \( t \in [0, \infty) \), then \( f \) is uniformly bounded throughout \([0, \infty)\).

Exercise 2. Show that the function
\[
H(x, y) = x^2 + y^2 + |x - y|^{-1}
\]
achieves its global minimum somewhere on the set \( \{(x, y) \in \mathbb{R}^2 : x \neq y\} \).

Exercise 3. Let \( a, b \in \mathbb{R} \) with \( a < b \) and let \( f : [a, b] \to [a, b] \) be continuous. Show that there exists \( x_0 \in [a, b] \) such that \( f(x_0) = x_0 \).

Exercise 4. Let \( f, g : [-1, 1] \to [-1, 1] \) be defined as follows:
\[
f(x) = \begin{cases} 
\frac{x}{2}, & x \in [-1, 0] \\
x - \frac{1}{2} \sin\left(\frac{x}{2}\right), & x \in (0, \frac{1}{2}] \\
x, & x \in \left[\frac{1}{2}, 1\right]
\end{cases}
\]
and
\[
g(x) = \begin{cases} 
\frac{1-x}{2}, & x \in [-1, 0] \\
-x - \frac{1}{2} \sin\left(\frac{x}{2}\right), & x \in (0, \frac{1}{2}] \\
-x, & x \in \left[\frac{1}{2}, 1\right].
\end{cases}
\]
Let \( A \) and \( B \) denote the graphs of \( f \) and \( g \), respectively, that is,
\[
A = \{(x, f(x)) : x \in [-1, 1]\} \quad \text{and} \quad B = \{(x, g(x)) : x \in [-1, 1]\}.
\]
(a) Show that \( A \cap B = \emptyset \) and \( \{(-1, -1), (1, 1)\} \subseteq A \) and \( \{(-1, 1), (1, -1)\} \subseteq B \).
(b) Prove that \( A \) and \( B \) are connected subsets of \([-1, 1] \times [-1, 1]\).

Exercise 5. Define \( f : \mathbb{R} \to \mathbb{R} \) by
\[
f(x) = \begin{cases} 
0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\
\frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ with } (p, q) = 1.
\end{cases}
\]
Prove that \( f \) is continuous on \( \mathbb{R} \setminus \mathbb{Q} \) and discontinuous on \( \mathbb{Q} \).

Exercise 6. Let \((X, d)\) be a metric space and let \( f, g : X \to \mathbb{R} \) be two continuous functions.

a) Prove that the set \( \{x \in X : f(x) < g(x)\} \) is open.

b) Prove that if the set \( \{x \in X : f(x) \leq g(x)\} \) is dense in \( X \), then \( f(x) \leq g(x) \) for all \( x \in X \).

Exercise 7. Let \( a, b \in \mathbb{R} \) with \( a < b \). Show that a function \( f : (a, b) \to \mathbb{R} \) is uniformly continuous on \((a, b)\) if and only if it can be extended to a continuous function \( \tilde{f} \) on \([a, b]\).