Exercise 1. Let \( \{A_i\}_{i \in I} \) be an infinite family of closed sets with the finite intersection property. Assuming that one member of this family is compact, show that \( \bigcap_{i \in I} A_i \neq \emptyset \).

Exercise 2. Let \((X, d)\) be a metric space and let \( A \subseteq X \) be a compact subset. Show that
(a) For any \( y \in X \) there exists \( a \in A \) so that \( d(y, A) = d(y, a) \).
(b) If \( B \subseteq X \) and \( d(A, B) = \inf \{d(a, b) : a \in A \text{ and } b \in B\} = 0 \) then \( A \cap B \neq \emptyset \).

Exercise 3. Let \((X, d)\) be a metric space. If \( A \) and \( B \) are two subsets of \( X \), we define their Hausdorff distance via
\[
d_H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \right\}.
\]
Let \( F(X) = \{A \subseteq X : A \text{ is compact and non-empty}\} \).
Show that
(a) \((F(X), d_H)\) is a metric space.
(b) If \((X, d)\) is compact, then so is \((F(X), d_H)\).

Exercise 4. Let \((X, d_X)\) be a compact metric space.
(a) Verify that \( d_Y(f, g) = \sum_{n \in \mathbb{Z}} 2^{-|n|} d_X(f(n), g(n)) \) defines a metric on \( Y = \{f : \mathbb{Z} \to X\} \).
(b) Show that \( Y \) is compact.

Exercise 5. (a) Show that the closed unit ball in \( \ell^2 \), namely,
\[
A = \{x \in \ell^2 : \sum_{n=1}^{\infty} |x_n|^2 \leq 1\}
\]
is not compact in \( \ell^2 \).
(b) Define \( B \subseteq \ell^2 \) by
\[
B = \{x \in \ell^2 : \sum_{n=1}^{\infty} n|x_n|^2 \leq 1\}.
\]
Show that \( B \) is compact.

Exercise 6. Let \( A \) be a subset of a complete metric space. Assume that for all \( \varepsilon > 0 \), there exists a compact set \( A_\varepsilon \) so that
\[
\forall x \in A, \quad d(x, A_\varepsilon) < \varepsilon.
\]
Show that \( \tilde{A} \) is compact.

Exercise 7. Let \((X, d_1)\) and \((Y, d_2)\) be two compact metric spaces. Show that the space \( X \times Y \) endowed with the ‘Euclidean’ distance
\[
d((x_1, y_1), (x_2, y_2)) = \sqrt{[d_1(x_1, x_2)]^2 + [d_2(y_1, y_2)]^2}
\]
is a compact metric space.
Exercise 8. Show that a totally bounded metric space contains a countable dense subset.

Exercise 9. Consider the Cantor set

\[ K = \{ x \in [0, 1] : x = \sum_{n=1}^{\infty} a_n 3^{-n} \text{ with all } a_n \in \{0, 2\} \}. \]

For example, 1 \in K because it is represented by setting all \( a_n = 2 \).

(a) Show that K is compact.
(b) Show that K is uncountable.
(c) Show that no connected subset of K contains more than one point.