HOMEWORK 10

Exercise 1. Assume $f, g : [a, b] \to \mathbb{R}$ are continuous functions such that $g(x) \geq 0$ for all $x \in [a, b]$. Show that there exists $x_0 \in [a, b]$ such that

$$\int_a^b f(t)g(t)\,dt = f(x_0)\int_a^b g(t)\,dt.$$

Exercise 2. Let $\{f_n\}_{n\geq 1}$ be a uniformly bounded sequence of functions that are Riemann integrable on $[a, b]$. For $n \geq 1$ we define $F_n : [a, b] \to \mathbb{R}$ via

$$F_n(x) = \int_a^x f_n(t)\,dt.$$

Prove that there exists a subsequence of $\{F_n\}_{n\geq 1}$ that converges uniformly on $[a, b]$.

Exercise 3. Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable function such that f'' is Riemann integrable on $[a, b]$.

(a) Show that

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\big[f(a) + f(b)\big] + \frac{1}{2}\int_a^b f''(x)(x-a)(x-b)\,dx.$$

(b) If additionally f'' is continuous, show that there exists $x_0 \in [a, b]$ such that

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\big[f(a) + f(b)\big] - \frac{(b-a)^3}{12}f''(x_0).$$

Exercise 4. Let $f : [0, 1] \to \mathbb{R}$ be a Riemann integrable function. Show that

$$\lim_{n\to\infty}\int_0^1 f(x)\sin(nx)\,dx = 0.$$

Exercise 5. For $n \geq 1$, let $f_n : [0, 1] \to \mathbb{R}$ be a continuous function satisfying

$$|f_n(x)| \leq 1 + \frac{n}{1 + n^2x^2}$$

and define $F_n : [0, 1] \to \mathbb{R}$ via

$$F_n(x) = \int_0^x f_n(t)\,dt.$$

Show that the sequence $\{F_n\}_{n\geq 1}$ admits a subsequence that converges pointwise on $[0, 1]$.

Exercise 6. Let $f : [0, 1] \to \mathbb{R}$ and $g : [0, 1] \to [0, 1]$ be two Riemann integrable functions. Assume that

$$|g(x) - g(y)| \geq \alpha|x - y| \quad \text{for any} \quad x, y \in [0, 1]$$

and some fixed $\alpha \in (0, 1)$. Show that $f \circ g$ is Riemann integrable.

Exercise 7. For $x \in (0, \infty)$, let

$$F(x) = \int_0^\infty \frac{1 - e^{-tx}}{t^{\frac{3}{2}}} \, dt.$$

Show that $F : (0, \infty) \to (0, \infty)$ is well-defined, bijective, of class C^1 (i.e. differentiable with continuous derivative), and that its inverse is of class C^1.