HOMEWORK 1

Exercise 1. Let A and B be two non-empty subsets of $\mathbb{R}$ that are bounded below. Let

$$S = \big\{a + b : \ a \in A \text{ and } b \in B\big\}.$$

Prove that

$$\inf S = \inf A + \inf B.$$

Exercise 2. Let X be the subset of l^∞ consisting of sequences of real numbers that have only finitely many non-zero entries:

$$X = \Big\{\{x_n\}_{n\geq 1} \subset \mathbb{R} : \ x_n \neq 0 \text{ for only finitely many } n \geq 1\Big\}.$$

We equip X with the d_∞ metric: for any two points $x = \{x_n\}_{n\geq 1}$ and $y = \{y_n\}_{n\geq 1}$ in X, we define

$$d_\infty(x, y) = \sup_{n\geq 1} |x_n - y_n|.$$

(a) Show that the sequence $\{x^{(k)}\}_{k\geq 1} \subset X$ given by

$$x^{(k)} = \Big(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k}, 0, 0, 0, \dots\Big)$$

is a Cauchy sequence.

(b) Conclude that (X, d_∞) is not a complete metric space.

Exercise 3. Let A be a non-empty set such that there exists an injective function $f : A \to A$ that is *not* surjective. Construct an injective function $g : \mathbb{N} \to A$.

Exercise 4. Let (X, d) be a metric space with X being a countable set. Show that X is not connected.

Exercise 5. Consider the metric space $(\mathbb{Q}, d)$ where $d(x, y) = |x - y|$. Let

$$A = \{r \in \mathbb{Q} : \ \sqrt{2} < r < \sqrt{3}\}.$$

(a) Show that A is open in $\mathbb{Q}$.

(b) Show that A is closed in $\mathbb{Q}$.

(c) Show that A does not have the Baire property.

Exercise 6. Consider $\mathbb{R}^2$ endowed with the Euclidean metric d_2. Let A be a non-empty closed and bounded subset of $\mathbb{R}^2$. Show that the set

$$S = \{x + y : \ (x, y) \in A\}$$

is a closed and bounded subset of $(\mathbb{R}, |\cdot|)$.

Exercise 7. Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$d\big((x_1, x_2), (y_1, y_2)\big) = \begin{cases} \sqrt{x_1^2 + x_2^2} + \sqrt{y_1^2 + y_2^2} & \text{if } (x_1, x_2) \neq (y_1, y_2) \\ 0 & \text{if } (x_1, x_2) = (y_1, y_2). \end{cases}$$

(a) Show that d is a metric.

(b) Show that $(\mathbb{R}^2, d)$ is not a connected metric space.

Exercise 8. Suppose $\{a_n\}_{n\geq 1}$ is a sequence of non-negative real numbers such that $s = \sum_{n\geq 1} a_n < \infty$. For $k \geq 1$, let N_k denote the cardinality of the set $\{n \in \mathbb{N} : a_n \geq 2^{-k}\}$. Show that

$$\limsup_{k\to\infty} 2^{-k} N_k = 0.$$