HOMEWORK 1

**Exercise 1.** Let $A$ and $B$ be two non-empty subsets of $\mathbb{R}$ that are bounded below. Let

$$S = \{a + b : a \in A \text{ and } b \in B\}.$$ 

Prove that

$$\inf S = \inf A + \inf B.$$ 

**Exercise 2.** Let $X$ be the subset of $l^\infty$ consisting of sequences of real numbers that have only finitely many non-zero entries:

$$X = \left\{\{x_n\}_{n \geq 1} \subset \mathbb{R} : x_n \neq 0 \text{ for only finitely many } n \geq 1\right\}.$$ 

We equip $X$ with the $d_\infty$ metric: for any two points $x = \{x_n\}_{n \geq 1}$ and $y = \{y_n\}_{n \geq 1}$ in $X$, we define

$$d_\infty(x, y) = \sup_{n \geq 1} |x_n - y_n|.$$ 

(a) Show that the sequence $\{x^{(k)}\}_{k \geq 1} \subset X$ given by

$$x^{(k)} = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{k}, 0, 0, 0, \ldots\right)$$

is a Cauchy sequence.

(b) Conclude that $(X, d_\infty)$ is not a complete metric space.

**Exercise 3.** Let $A$ be a non-empty set such that there exists an injective function $f : A \to A$ that is not surjective. Construct an injective function $g : \mathbb{N} \to A$.

**Exercise 4.** Let $(X, d)$ be a metric space with $X$ being a countable set. Show that $X$ is not connected.

**Exercise 5.** Consider the metric space $(\mathbb{Q}, d)$ where $d(x, y) = |x - y|$. Let

$$A = \{r \in \mathbb{Q} : \sqrt{2} < r < \sqrt{3}\}.$$ 

(a) Show that $A$ is open in $\mathbb{Q}$.

(b) Show that $A$ is closed in $\mathbb{Q}$.

(c) Show that $A$ does not have the Baire property.

**Exercise 6.** Consider $\mathbb{R}^2$ endowed with the Euclidean metric $d_2$. Let $A$ be a non-empty closed and bounded subset of $\mathbb{R}^2$. Show that the set

$$S = \{x + y : (x, y) \in A\}$$

is a closed and bounded subset of $(\mathbb{R}, |\cdot|)$.

**Exercise 7.** Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$d((x_1, x_2), (y_1, y_2)) = \begin{cases} \sqrt{x_1^2 + x_2^2} + \sqrt{y_1^2 + y_2^2} & \text{if } (x_1, x_2) \neq (y_1, y_2) \\ 0 & \text{if } (x_1, x_2) = (y_1, y_2). \end{cases}$$ 

(a) Show that $d$ is a metric.

(b) Show that $(\mathbb{R}^2, d)$ is not a connected metric space.
Exercise 8. Suppose \( \{a_n\}_{n \geq 1} \) is a sequence of non-negative real numbers such that \( s = \sum_{n \geq 1} a_n < \infty \). For \( k \geq 1 \), let \( N_k \) denote the cardinality of the set \( \{ n \in \mathbb{N} : a_n \geq 2^{-k} \} \). Show that

\[
\limsup_{k \to \infty} 2^{-k} N_k = 0.
\]