HOMEWORK 9

Exercise 1. Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function such that $f \geq 0$ and

$$\int_a^b f(x)\,dx = 0.$$

Show that if $x \in [a, b]$ is a point of continuity for f then $f(x) = 0$.

Exercise 2. Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function such that

$$\int_a^b x^n f(x)\,dx = 0 \quad \text{for all} \quad n \geq 0.$$

Show that if $x \in [a, b]$ is a point of continuity for f then $f(x) = 0$.

Exercise 3. Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable functions such that g is monotone. Show that there exists $x_0 \in [a, b]$ such that

$$\int_a^b f(x)g(x)\,dx = g(a)\int_a^{x_0} f(x)\,dx + g(b)\int_{x_0}^b f(x)\,dx.$$

Hint: Show that if g is monotonically decreasing on $[a, b]$ with $g(b) = 0$, then

$$g(a)\inf_{x\in[a,b]}\int_a^x f(t)\,dt \leq \int_a^b f(x)g(x)\,dx \leq g(a)\sup_{x\in[a,b]}\int_a^x f(t)\,dt.$$

Exercise 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and define $F : \mathbb{R} \to \mathbb{R}$ via

$$F(x) = \int_{x-1}^{x+1} f(t)\,dt.$$

Show that F is differentiable and compute its derivative.

Exercise 5. Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable function such that f'' is Riemann integrable on $[a, b]$.

(a) Show that

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\big[f(a) + f(b)\big] + \frac{1}{2}\int_a^b f''(x)(x-a)(x-b)\,dx.$$

(b) If additionally f'' is continuous, show that there exists $x_0 \in [a, b]$ such that

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\big[f(a) + f(b)\big] - \frac{(b-a)^3}{12}f''(x_0).$$

Exercise 6. For $n \geq 1$, let $f_n : [a, b] \to \mathbb{R}$ be a continuous function. Assume that f_n converges pointwise to a continuous function $f : [a, b] \to \mathbb{R}$. Assume that there exists $M > 0$ such that

$$|f_n(x)| \leq M \quad \text{for all } x \in [a, b] \text{ and all } n \geq 1.$$

Show that

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Exercise 7. For $n \geq 1$, let $f_n : [0,1] \to \mathbb{R}$ be a continuous function satisfying

$$|f_n(x)| \leq 1 + \frac{n}{1+n^2x^2}$$

and define $F_n : [0,1] \to \mathbb{R}$ via

$$F_n(x) = \int_0^x f_n(t)\,dt.$$

Show that the sequence $\{F_n\}_{n\geq 1}$ admits a subsequence that converges pointwise on $[0,1]$.