HOMEWORK 8

Exercise 1. Show that if $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then f is bounded on $[a, b]$.

Exercise 2. Let $f : [a, b] \to \mathbb{R}$ be integrable and let $g : [a, b] \to \mathbb{R}$ be a function such that $f(x) = g(x)$ except for finitely many x in $[a, b]$. Show that g is integrable on $[a, b]$ and that

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx.$$

Exercise 3. Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$.

(a) Show that if P is a partition of $[a, b]$, then

$$U(f^2, P) - L(f^2, P) \leq 2M\big[U(f, P) - L(f, P)\big].$$

(b) Deduce that if f is integrable on $[a, b]$, then f^2 is also integrable on $[a, b]$.

(c) Prove that if f and g are two integrable functions on $[a, b]$, then the product fg is integrable on $[a, b]$.

Exercise 4. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(x) \geq 0$ for all $x \in [a, b]$. Assume that

$$\int_a^b f(x)\,dx = 0.$$

Show that $f(x) = 0$ for all $x \in [a, b]$.

Exercise 5. Let $f, g : [a, b] \to \mathbb{R}$ be two Riemann integrable functions such that the set $\{x \in [a, b] : f(x) = g(x)\}$ is dense in $[a, b]$. Show that

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx.$$

Exercise 6. Suppose $f : [1, \infty) \to \mathbb{R}$ is Riemann integrable on $[1, a]$ for all $a > 1$. If

$$\lim_{a\to\infty} \int_1^a f(x)\,dx$$

exists and is finite, we say the integral $\int_1^\infty f(x)\,dx$ converges and we write

$$\int_1^\infty f(x)\,dx = \lim_{a\to\infty} \int_1^a f(x)\,dx.$$

Now assume $f : [1, \infty) \to \mathbb{R}$ is non-negative and decreasing. Show that

$$\int_1^\infty f(x)\,dx \text{ converges if and only if } \sum_{n\geq 1} f(n) \text{ converges.}$$