HOMEWORK 7

Exercise 1. Let $f:[0,1]\to\mathbb{R}$. We say that f is Hölder continuous of order $\alpha\in(0,1)$ and write $f\in C^\alpha([0,1])$ if

$$\|f\|_{C^\alpha}=\sup\{|f(x)|:x\in[0,1]\}+\sup\left\{\tfrac{|f(x)-f(y)|}{|x-y|^\alpha}:x,y\in[0,1]\text{ with }x\neq y\right\}<\infty.$$

Let $d_\alpha: C^\alpha([0,1])\times C^\alpha([0,1])\to\mathbb{R}$ be given by

$$d_\alpha(f,g)=\|f-g\|_{C^\alpha}.$$

(a) Show that $(C^\alpha([0,1]),d_\alpha)$ is a complete metric space.

(b) Show that any bounded sequence in $(C^{1/2}([0,1]),d_{1/2})$ admits a subsequence that converges in $(C^{1/3}([0,1]),d_{1/3})$.

Exercise 2. Let $f:\mathbb{R}\to\mathbb{R}$ be a Lipschitz function, that is,

$$\sup\left\{\tfrac{|f(x)-f(y)|}{|x-y|}:x,y\in\mathbb{R}\text{ with }x\neq y\right\}<\infty.$$

Suppose that for every $x\in\mathbb{R}$,

$$\lim_{n\to\infty}n\big[f\big(x+\tfrac{1}{n}\big)-f(x)\big]=\lim_{n\to\infty}n\big[f\big(x-\tfrac{1}{n}\big)-f(x)\big]=0.$$

Prove that f is differentiable on $\mathbb{R}$.

Exercise 3. Let $f:\mathbb{R}\to\mathbb{R}$ be a twice differentiable function such that

$$f(x)\geq 0\quad\text{and}\quad f''(x)\leq 0\quad\text{for all}\quad x\in\mathbb{R}.$$

Show that f is constant.

Exercise 4. Assume $f:(a,b)\to\mathbb{R}$ be a twice differentiable function. Show that for any $x\in(a,b)$, the limit

$$\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}$$

exists and equals $f''(x)$.

Exercise 5. We say that a function $f:[a,b]\to\mathbb{R}$ is a convex function if

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y)\quad\text{for all }x,y\in[a,b]\text{ and for all }t\in[0,1].$$

Show that for any $x\in(a,b)$ the limits

$$\lim_{y\searrow x}\frac{f(y)-f(x)}{y-x}\quad\text{and}\quad\lim_{y\nearrow x}\frac{f(y)-f(x)}{y-x}$$

exist and are finite.

Hint: Show that for all $a\leq x<y<z\leq b$ we have

$$\frac{f(y)-f(x)}{y-x}\leq\frac{f(z)-f(x)}{z-x}\leq\frac{f(z)-f(y)}{z-y}.$$