HOMEWORK 6

Exercise 1. Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Show that for any $c \in (a, b)$ that is not a point of maximum or minimum for f' there exist $x_1, x_2 \in (a, b)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Exercise 2. Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Assume that f' is strictly increasing. Show that for any $c \in (a, b)$ such that $f'(c) = 0$ there exist $x_1, x_2 \in [a, b]$, $x_1 < c < x_2$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Exercise 3. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function on the closed interval $[0, 1]$ and differentiable on the open interval $(0, 1)$. Assume that $f(0) = 0$ and f' is an increasing function on $(0, 1)$. Show that

$$g(x) = \frac{f(x)}{x}$$

is an increasing function on $(0, 1)$.

Exercise 4. Assume $f : [a, b] \to \mathbb{R}$ is a continuous function on the closed interval $[a, b]$ and differentiable on the open interval (a, b) with $f(a) = f(b) = 0$. Prove that for every $\lambda \in \mathbb{R}$ there exists $x_0 \in (a, b)$ such that $f'(x_0) = \lambda f(x_0)$.

Exercise 5. Assume $f : (1, \infty) \to \mathbb{R}$ is differentiable. If

$$\lim_{x \to \infty} f(x) = 1 \quad \text{and} \quad \lim_{x \to \infty} f'(x) = c,$$

prove that $c = 0$.