HOMEWORK 5

Exercise 1. Prove that a polynomial of degree n is uniformly continuous on $\mathbb{R}$ if and only if $n = 0$ or $n = 1$.

Exercise 2. Let $f : [0, 1] \to [0, 1]$ be a continuous function such that $f(0) = 0$ and $f(1) = 1$. Consider the sequence of functions $f_n : [0, 1] \to [0, 1]$ defined as follows:

$$f_1 = f \quad \text{and} \quad f_{n+1} = f \circ f_n \text{ for } n \geq 1.$$

Prove that if $\{f_n\}_{n\geq 1}$ converges uniformly, then $f(x) = x$ for all $x \in [0, 1]$.

Exercise 3. Let

$$\mathcal{F} = \big\{ f \in C(\mathbb{R}) : \lim_{|x|\to\infty} f(x) = 0 \big\}.$$

Show that $\mathcal{F}$ is closed in $C(\mathbb{R})$.

Exercise 4. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-x^2}$. Find
(a) an open set $D \subseteq \mathbb{R}$ such that $f(D)$ is not open;
(b) a closed set $F \subseteq \mathbb{R}$ such that $f(F)$ is not closed;
(c) a set $A \subseteq \mathbb{R}$ such that $f(\bar{A}) \neq \overline{f(A)}$.

Exercise 5. Let $\{F_n\}_{n\geq 1}$ be a sequence of closed sets such that $F_n \subseteq F_{n+1}$ for all $n \geq 1$. Set $F = \cup_{n\geq 1} F_n$ and $F_0 = \emptyset$. For $n \geq 1$ we define

$$A_n = \big[(F_n \setminus F_{n-1}) \setminus \operatorname{Int}(F_n \setminus F_{n-1})\big] \cup \big[\operatorname{Int}(F_n \setminus F_{n-1}) \cap \mathbb{Q}\big].$$

Let $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 2^{-n} & \text{if } x \in A_n \\ 0 & \text{if } x \notin \cup_{n\geq 1} A_n. \end{cases}$$

Show that f is discontinuous on F and continuous on $\mathbb{R} \setminus F$.
Remark: This exercise shows that given any F_σ subset of $\mathbb{R}$, there is a function whose set of discontinuities is precisely that set.

Exercise 6. Let (X, d) be a metric space with at least two points and let $\mathcal{A} \subseteq C(X)$ be an algebra that is dense in the metric space $C(X)$.

(a) Show that $\mathcal{A}$ separates points on X.

(b) Show that $\mathcal{A}$ vanishes at no point in X.

See pages 161-162 in Rudin's textbook for the definitions.

Exercise 7.
(a) Show that given any continuous function $f : [0, 1] \times [0, 1] \to \mathbb{R}$ and any $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and functions $g_1, \ldots g_n, h_1, \ldots, h_n \in C([0, 1])$ such that

$$\Big| f(x, y) - \sum_{k=1}^{n} g_k(x) h_k(y) \Big| < \varepsilon \quad \text{for all} \quad (x, y) \in [0, 1] \times [0, 1].$$

(b) If $f(x, y) = f(y, x)$ for all $(x, y) \in [0, 1] \times [0, 1]$, can this be done with $g_k = h_k$ for each $1 \leq k \leq n$? Justify your answer!