HOMEWORK 3

Exercise 1. Let $f : [0, \infty) \to [0, \infty)$ be continuous with $f(0) = 0$. Show that if
$$f(t) \le 1 + \tfrac{1}{10} f(t)^2 \qquad \text{for all } t \in [0, \infty),$$
then f is uniformly bounded throughout $[0, \infty)$.

Exercise 2. Show that the function
$$H(x, y) = x^2 + y^2 + |x - y|^{-1}$$
achieves its global minimum somewhere on the set $\{(x, y) \in \mathbb{R}^2 : x \ne y\}$.

Exercise 3. Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a, b] \to [a, b]$ be continuous. Show that there exists $x_0 \in [a, b]$ such that $f(x_0) = x_0$.

Exercise 4. Let $f, g : [-1, 1] \to [-1, 1]$ be defined as follows:
$$f(x) = \begin{cases} \frac{x-1}{2}, & x \in [-1, 0] \\ x - \frac{1}{2}\sin\left(\frac{\pi}{x}\right), & x \in (0, \frac{1}{2}] \\ x, & x \in [\frac{1}{2}, 1] \end{cases}$$
and
$$g(x) = \begin{cases} \frac{1-x}{2}, & x \in [-1, 0] \\ -x - \frac{1}{2}\sin\left(\frac{\pi}{x}\right), & x \in (0, \frac{1}{2}] \\ -x, & x \in [\frac{1}{2}, 1]. \end{cases}$$
Let A and B denote the graphs of f and g, respectively, that is,
$$A = \{(x, f(x)) : x \in [-1, 1]\} \quad \text{and} \quad B = \{(x, g(x)) : x \in [-1, 1]\}.$$
(a) Show that $A \cap B = \emptyset$ and $\{(-1, -1), (1, 1)\} \subseteq A$ and $\{(-1, 1), (1, -1)\} \subseteq B$.
(b) Prove that A and B are connected subsets of $[-1, 1] \times [-1, 1]$.

Exercise 5. Define $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ with } (p, q) = 1. \end{cases}$$
Prove that f is continuous on $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous on $\mathbb{Q}$.

Exercise 6. Let (X, d) be a metric space and let $f, g : X \to \mathbb{R}$ be two continuous functions.
a) Prove that the set $\{x \in X : f(x) < g(x)\}$ is open.
b) Prove that if the set $\{x \in X : f(x) \le g(x)\}$ is dense in X, then $f(x) \le g(x)$ for all $x \in X$.

Exercise 7. Let $a, b \in \mathbb{R}$ with $a < b$. Show that a function $f : (a, b) \to \mathbb{R}$ is uniformly continuous on (a, b) if and only if it can be extended to a continuous function $\tilde{f}$ on $[a, b]$.