HOMEWORK 2

Exercise 1. Let $\{A_i\}_{i\in I}$ be an infinite family of closed sets with the finite intersection property. Assuming that one member of this family is compact, show that $\cap_{i\in I} A_i \neq \varnothing$.

Exercise 2. Let (X,d) be a metric space and let $A \subseteq X$ be a compact subset. Show that
(a) For any $y \in X$ there exists $x \in A$ so that $d(y,A) = d(y,x)$.
(b) If $B \subseteq X$ and $d(A,B) = 0$ then $A \cap \bar{B} \neq \varnothing$.

Exercise 3. Let (X, d_X) be a compact metric space.
(a) Verify that

$$d_Y(f,g) = \sum_{n\in\mathbb{Z}} 2^{-|n|} d_X\big(f(n), g(n)\big)$$

defines a metric on $Y = \{f : \mathbb{Z} \to X\}$.
(b) Show that Y is compact.

Exercise 4. (a) Show that the closed unit ball in ℓ^2, namely,

$$A = \big\{x \in \ell^2 : \textstyle\sum_{n=1}^{\infty} |x_n|^2 \leq 1\big\}$$

is not compact in ℓ^2.
(b) Define $B \subseteq \ell^2$ by

$$B = \big\{x \in \ell^2 : \textstyle\sum_{n=1}^{\infty} n|x_n|^2 \leq 1\big\}.$$

Show that B is compact.

Exercise 5. Let A be a subset of a complete metric space. Assume that for all $\varepsilon > 0$, there exists a compact set A_ε so that

$$\forall x \in A, \quad d(x, A_\varepsilon) < \varepsilon.$$

Show that $\bar{A}$ is compact.

Exercise 6. Let (X, d_1) and (Y, d_2) be two compact metric spaces. Show that the space $X \times Y$ endowed with the 'Euclidean' distance

$$d\big((x_1,y_1),(x_2,y_2)\big) = \sqrt{d_1^2(x_1,x_2) + d_2^2(y_1,y_2)}$$

is a compact metric space.

Remark 1. The result in Exercise 6 continues to hold if the 'Euclidean' metric is replaced by either of the equivalent metrics

$$d\big((x_1,y_1),(x_2,y_2)\big) = d_1(x_1,x_2) + d_2(y_1,y_2)$$

or

$$d\big((x_1,y_1),(x_2,y_2)\big) = \max\{d_1(x_1,x_2), d_2(y_1,y_2)\}.$$

Exercise 7. Consider the Cantor set

$$K = \{x \in [0,1] : x = \sum_{n=1}^{\infty} a_n 3^{-n} \text{ with all } a_n \in \{0,2\}\}.$$

For example, $1 \in K$ because it is represented by setting all $a_n = 2$.

(a) Show that K is compact.

(b) Show that K is uncountable.

(c) Show that no connected subset of K contains more than one point.