HOMEWORK 10

Exercise 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \frac{|x|^{\alpha_1}|y|^{\alpha_2}}{(|x|^{\beta_1}+|y|^{\beta_2})^{\gamma}} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0), \end{cases}$$

where $\alpha_1, \alpha_2 \geq 0$ and $\beta_1, \beta_2, \gamma > 0$. Prove that f is continuous if and only if

$$\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} > \gamma.$$

Exercise 2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} (x^2+y^2)^{\alpha} & \text{if} \quad x \in \mathbb{Q} \text{ and } y \in \mathbb{Q} \\ 0 & \text{if} \quad x \notin \mathbb{Q} \text{ or } y \notin \mathbb{Q}. \end{cases}$$

Show that f is differentiable at $(0,0)$ if and only if $\alpha > \frac{1}{2}$.

Exercise 3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$$

1) Show that f is differentiable on $\mathbb{R}^2 \setminus \{(0,0)\}$, but f is not differentiable at $(0,0)$.

2) Show that for any unit vector $u \in \mathbb{R}^2$, f is differentiable at $(0,0)$ in the direction u.

Exercise 4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \frac{y^5}{x^6+y^4} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$$

Study the differentiability of f and the continuity of its partial derivatives.

Exercise 5. Let G be the open set in $\mathbb{R}^2$ defined via

$$G = \{(x,y) \in \mathbb{R}^2 : (x,y) \notin [0,\infty) \times \{0\}\}.$$

Let $f : G \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} x^3 & \text{if} \quad x > 0 \text{ and } y > 0 \\ 0 & \text{if} \quad x \leq 0 \text{ or } y \leq 0. \end{cases}$$

1) Show that f is differentiable at every point in G.

2) Show that

$$\frac{\partial f}{\partial y}(x,y) = 0 \quad \text{for all} \quad (x,y) \in G$$

and yet the function $y \mapsto f(x,y)$ is not constant.

Exercise 6. Let G be an open subset of $\mathbb{R}^n$ and assume $f : G \to \mathbb{R}$ is differentiable on G and has a local maximum at a point $a \in G$. Prove that $f'(a) = 0$.

Exercise 7. Show that the system of equations

$$\begin{cases} x^2 + y^2 z + z^3 = 1 \\ xz^2 + y^3 = -1 \end{cases}$$

defines in a neighborhood of $(1, -1, 0)$ a unique explicit function $\phi(x) = (y, z)$, which is twice continuously differentiable. Compute $\phi'(1)$ and $\phi''(1)$.