HOMEWORK 1

Exercise 1. Let $(F, +, \cdot, <)$ be an ordered field with at least two elements and let 1 denote the identity for multiplication. Show that the equation

$$x^2 = 1$$

has exactly two solutions in F.

Exercise 2. Fix $n \geq 1$. Show that the set of all subsets of $\mathbb{N}$ with n elements is countable.

Exercise 3. Let (X, d) be a metric space. Let A be a subset of X and let A' denote the set of accumulation points of A. Show that A' is closed.

Exercise 4. Let A and B be two non-empty sets of real numbers. Assume that A is open. Prove that the set

$$A + B = \{a + b \,:\, a \in A \text{ and } b \in B\}$$

is open.

Exercise 5. Consider the complete metric space (ℓ^∞, d_∞) defined as follows:

$$\ell^\infty = \big\{\{x_n\}_{n\geq 1} \subseteq \mathbb{R} : \sup_{n\geq 1} |x_n| < \infty\big\}$$

and for two points $x = \{x_n\}_{n\geq 1} \in \ell^\infty$ and $y = \{y_n\}_{n\geq 1} \in \ell^\infty$, the distance is given by

$$d_\infty(x, y) = \sup_{n\geq 1} |x_n - y_n|.$$

Let

$$A = \big\{x \in \ell^\infty : \sup_{n\geq 1} n|x_n| \leq 1\big\}.$$

Show that A is a closed subset of ℓ^∞.

Exercise 6. Let (X, d) be a metric space. For $a, b \in X$ we write $a \sim b$ if there exists a connected subset A of X such that $\{a, b\} \subseteq A$.

(a) Show that $\sim$ defines an equivalence relation on X.

(b) For $a \in X$, let C_a denote the equivalence class of a, that is,

$$C_a = \{b \in X \,:\, a \sim b\}.$$

Show that C_a is a connected set. Show that C_a is the largest connected subset of X that contains a.

(c) Given $a \in X$, show that C_a is a closed set.

(d) Given $a, b \in X$ such that a is not equivalent to b, show that C_a and C_b are separated sets.

Exercise 7. Suppose A and B are compact subsets of a metric space. Show that $A \cap B$ and $A \cup B$ are also compact.