HOMEWORK 1

Exercise 1. (5 points) Using the truth table, show that $A \Rightarrow B$ is logically equivalent to (not B) $\Rightarrow$ (not A).

Exercise 2. (10 points) Using the truth table, verify de Morgan's laws, namely,

$$\text{not } (A \text{ and } B) = (\text{not } A) \text{ or } (\text{not } B)$$
$$\text{not } (A \text{ or } B) = (\text{not } A) \text{ and } (\text{not } B)$$

Exercise 3. (15 points) Negate the following sentences:
- If pigs had wings, they would fly.
- If that plane leaves and you are not on it, then you will regret it.
- For every problem there is a solution that is neat, plausible, and wrong.

Exercise 4. (5 points) Let X and Y be statements. If we want to DISPROVE the claim that "At least one of X and Y are true", which one of the following do we need to show?
a) At least one of X and Y are false.
b) X and Y are both false.
c) Exactly one of X and Y are false.
d) Y is false.
e) X does not imply Y, and Y does not imply X.
f) X is true if and only if Y is false.
g) X is false.

Exercise 5. (5 points) Let X and Y be statements. If we know that X implies Y, which one of the following can we conclude?
a) X cannot be false.
b) X is true and Y is also true.
c) If Y is false, then X is false.
d) Y cannot be false.
e) If X is false, then Y is false.
f) If Y is true, then X is true.
g) At least one of X and Y is true.

Exercise 6. (5 points) Let X,Y,Z be statements. Suppose we know that "X is true implies Y is true", and "X is false implies Z is true". If we know that Z is false, then which one of the following can we conclude?
a) X is false.
b) X is true.
c) Y is true.
d) b) and c).
e) a) and c).
f) a), b), and c).
g) None of the above conclusions can be drawn.

Exercise 7. (5 points) Let X and Y be statements. Which of the following strategies is NOT a valid way to show that "X implies Y"?

a) Assume that X is true and then use this to show that Y is true.
b) Show that X implies some intermediate statement Z and then show that Z implies Y.
c) Assume that X is true, and Y is false, and deduce a contradiction.
d) Assume that Y is false and then use this to show that X is false.
e) Show that some intermediate statement Z implies Y and then show that X implies Z.
f) Assume that X is false, and Y is true, and deduce a contradiction.
g) Show that either X is false, or Y is true, or both.

Exercise 8. (5 points) Let $P(x)$ be a property about some object x of type X. If we want to DISPROVE the claim that "$P(x)$ is true for all x of type X", which one of the following do we have to do?
a) Show that for every x in X, $P(x)$ is false.
b) Show that for every x in X, there is a y not equal to x for which $P(y)$ is true.
c) Show that $P(x)$ being true does not necessarily imply that x is of type X.
d) Show that there are no objects x of type X.
e) Show that there exists an x of type X for which $P(x)$ is false.
f) Show that there exists an x which is not of type X, but for which $P(x)$ is still true.
g) Assume there exists an x of type X for which $P(x)$ is true, and derive a contradiction.

Exercise 9. (5 points) Let $P(n, m)$ be a property about two integers n and m. If we want to DISPROVE the claim that "There exists an integer n such that $P(n, m)$ is true for all integers m", then which one of the following do we need to prove?
a) If $P(n, m)$ is true, then n and m are not integers.
b) For every integer n, there exists an integer m such that $P(n, m)$ is false.
c) For every integer n, and every integer m, the property $P(n, m)$ is false.
d) For every integer m, there exists an integer n such that $P(n, m)$ is false.
e) There exists an integer n such that $P(n, m)$ is false for all integers m.
f) There exists integers n, m such that $P(n, m)$ is false.
g) There exists an integer m such that $P(n, m)$ is false for all integers n.

Exercise 10. (10 points) Prove that

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

for all natural numbers n.

Exercise 11. (10 points) Prove that $2^{n+2} > 2n + 5$ for all natural numbers n.

Exercise 12. (10 points) Decide for which natural numbers the inequality $3^n > n^3$ is true. Prove your claim using mathematical induction.

Exercise 13. (10 points) Prove that $n^5 - n$ is divisible by 30 for all natural numbers n.