HOMEWORK 9

Exercise 1. Let X be a non-empty set and let $d : X \times X \to \mathbb{R}$ be the discrete metric on X defined as follows: for any $x, y \in X$,

$$d(x,y) = \begin{cases} 0, & \text{if} \quad x = y \\ 1, & \text{if} \quad x \neq y. \end{cases}$$

Find the open and the closed subsets of this metric space.

Exercise 2. Let (X, d_1) be a metric space and let $d_2 : X \times X \to \mathbb{R}$ be the metric defined as follows: for any $x, y \in X$,

$$d_2(x,y) = \frac{d_1(x,y)}{1 + d_1(x,y)}.$$

Prove that a subset A of X is open with respect to the metric d_1 if and only if it is open with respect to the metric d_2.

Exercise 3. Let $1 \leq p, q \leq \infty$ and consider the two metrics on $\mathbb{R}^n$ given by

$$d_p(x,y) = \Big(\sum_{k=1}^{n} |x_k - y_k|^p\Big)^{1/p} \quad \text{and} \quad d_q(x,y) = \Big(\sum_{k=1}^{n} |x_k - y_k|^q\Big)^{1/q},$$

with the usual convention if p or q are infinity. Prove that a set $A \subseteq \mathbb{R}^n$ is open with respect to the metric d_p if and only if it is open with respect to the metric d_q.

Exercise 4. Let (X, d) be a metric space and let A be a non-empty subset of X. Prove that A is open if and only if it can be written as the union of a family of open balls of the form $B_r(x) = \{y \in X : d(x,y) < r\}$.

Exercise 5. Let (X, d) be a metric space. The diameter of a set $\emptyset \neq A \subseteq X$ is given by

$$\delta(A) = \sup\{d(x,y) : x, y \in A\},$$

with the convention that $\delta(A) = \infty$ if the set $\{d(x,y) : x, y \in A\}$ is unbounded.

Prove that the diameter of A is equal to the diameter of the closure of A, that is, $\delta(A) = \delta(\bar{A})$.

Exercise 6. Fix $r > 0$. Let (X, d) be a metric space and let A be a non-empty subset of X with diameter $\delta(A) < r$. Let $a \in X$ and assume that $A \cap B_r(a) \neq \emptyset$. Then $A \subseteq B_{2r}(a)$.

Exercise 7. Let (X, d) be a metric space and let A be a subset of X and O be an open subset of X. Prove that

$$O \cap \bar{A} \subseteq \overline{O \cap A} \quad \text{and} \quad \overline{O \cap \bar{A}} = \overline{O \cap A}.$$

Conclude that if $O \cap A = \emptyset$, then $O \cap \bar{A} = \emptyset$.

Definition 0.1. Let (X, d) be a metric space. The *frontier* of a set $A \subseteq X$ is given by

$$\mathrm{Fr}(A) = \bar{A} \cap \overline{{}^{c}A}$$

Exercise 8. Let (X, d) be a metric space and let A be a subset of X. Prove that A is closed if and only if $\mathrm{Fr}(A) \subseteq A$.

Exercise 9. Let (X, d) be a metric space and let A be a subset of X. Prove that A is open if and only if $\mathrm{Fr}(A) \cap A = \emptyset$.

Exercise 10. Let (X, d) be a metric space and let A, B be two subsets of X. Prove that

$$\mathrm{Fr}(A \cup B) \subseteq \mathrm{Fr}(A) \cup \mathrm{Fr}(B).$$

Show also that if $\bar{A} \cap \bar{B} = \emptyset$, then $\mathrm{Fr}(A \cup B) = \mathrm{Fr}(A) \cup \mathrm{Fr}(B)$.

Exercise 11. Let $F \subseteq \mathbb{R}$ be a closed set bounded above. Prove that the least upper bound of F belongs to F.

Exercise 12. Let $\mathbb{R}^n$ be endowed with the Euclidean metric d_2. Let S be a non-empty subset of $\mathbb{R}^n$; in particular, $(S, d_2|_{S \times S})$ is a metric space.
1) Given $x \in S$, is the set $\{y \in S : d_2(x, y) \geq r\}$ closed in S?
2) Given $x \in S$, is the set $\{y \in S : d_2(x, y) \geq r\}$ contained in the closure of $\{y \in S : d_2(x, y) > r\}$ in S?