HOMEWORK 8

Exercise 1. Fix $n \geq 1$. Show that if $A_1, A_2, \ldots, A_n$ are countable sets, then the cartesian product $A_1 \times A_2 \times \ldots \times A_n$ is countable.

Exercise 2. If the sets A and B are equipotent ($A \sim B$), show that $\mathcal{P}(A) \sim \mathcal{P}(B)$.

Exercise 3. Prove that $\mathcal{P}(\mathbb{N})$ is equipotent with the set of functions

$$2^{\mathbb{N}} = \{f : \mathbb{N} \to \{0, 1\} : f \text{ is a function}\}.$$

In particular, the cardinality of $\mathcal{P}(\mathbb{N})$ is $2^{\aleph_0}$.

Exercise 4. Show that $\mathbb{N}^{\mathbb{N}} \sim 2^{\mathbb{N}}$, that is, the set of sequences with values in $\mathbb{N}$ is equipotent with the set of sequences with values in $\{0, 1\}$.

Exercise 5. Fix $n \geq 1$ and let $\mathcal{P}$ denote the set of polynomials of degree n with integer coefficients, that is,

$$\mathcal{P} = \{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 : \ a_i \in \mathbb{Z} \text{ for all } 1 \leq i \leq n \text{ and } a_n \neq 0\}.$$

Show that the set of all real roots of all polynomials in $\mathcal{P}$, that is,

$$\mathcal{A} = \{x \in \mathbb{R} : \ \text{there exists } p \in \mathcal{P} \text{ such that } p(x) = 0\}$$

is countable.

Exercise 6. Fix $n \geq 1$. Show that the set of all subsets of $\mathbb{N}$ with n distinct elements is countable.

Exercise 7. Show that the cardinality of $\mathbb{R}$ is $2^{\aleph_0}$. You may use the fact that the interval $(0, 1)$ has cardinality $2^{\aleph_0}$.

Exercise 8. Prove that the set of irrational numbers has the cardinality of $\mathbb{R}$.