HOMEWORK 5

Exercise 1. Let

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = \left[1 - \tfrac{1}{(n+1)^2}\right] a_n \text{ for all } n \geq 1.$$

1) Show that the sequence $\{a_n\}_{n\geq 1}$ converges.

2) Find its limit.

Exercise 2. Let A be a non-empty bounded subset of $\mathbb{R}$ and suppose $\sup A \notin A$. Show that there exists an increasing sequence of points $\{a_n\}_{n\geq 1}$ in A such that $\lim_{n\to\infty} a_n = \sup A$.

Exercise 3. Let $\mathcal{C}$ be the set of Cauchy sequences of rational numbers. Define the relation $\sim$ as follows: if $\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1} \in \mathcal{C}$, we write $\{a_n\}_{n\geq 1} \sim \{b_n\}_{n\geq 1}$ if the sequence $\{a_n - b_n\}_{n\geq 1}$ converges to zero.

1) Prove that $\sim$ is an equivalence relation on $\mathcal{C}$.

2) For $\{a_n\}_{n\geq 1} \in \mathcal{C}$, we denote its equivalence class by $[a_n]$. Let F denote the set of equivalence classes in $\mathcal{C}$. We define addition and multiplication on F as follows:

$$[a_n] + [b_n] = [a_n + b_n] \quad \text{and} \quad [a_n] \cdot [b_n] = [a_n b_n].$$

Show that these internal laws of composition are well defined and that F together with these operations is a field.

3) We define a relation on F as follows: we write $[a_n] < [b_n]$ if $[a_n] \neq [b_n]$ and there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $a_n < b_n$.
Prove that this relation is well defined.
Show that the set of positive elements in F, that is,

$$P = \{[a_n] \in F : [a_n] > 0\}$$

satisfies the following properties:
01') For every $[a_n] \in F$, exactly one of the following holds:

$$[a_n] = [0], \quad [a_n] \in P, \qquad -[a_n] \in P$$

where $[0]$ denotes the equivalence class of the sequence identically equal to zero.
02') For every $[a_n], [b_n] \in P$, we have $[a_n] + [b_n] \in P$ and $[a_n] \cdot [b_n] \in P$.

4) Conclude that F is an ordered field.

Caution: For Exercise 3, you may not use that Cauchy sequences converge in $\mathbb{R}$ (which is a consequence of the fact that $\mathbb{R}$ has the least upper bound property). The purpose of the exercise is to provide another construction of an ordered field with the least upper bound property. The fact that F in Exercise 3 has the least upper bound property is somewhat more involved and not assigned in the homework.

Exercise 4. Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two bounded sequences. Show that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Exercise 5. Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two bounded sequences of non-negative numbers. Show that

$$\limsup_{n\to\infty}(a_n b_n) \leq \big(\limsup_{n\to\infty} a_n\big)\big(\limsup_{n\to\infty} b_n\big).$$

Exercise 6. Show that a sequence $\{a_n\}_{n\geq 1}$ is bounded if and only if $\limsup |a_n| < \infty$.

Exercise 7. Let A denote the set of subsequential limits of a sequence $\{a_n\}_{n\geq 1}$. Suppose that $\{b_n\}_{n\geq 1}$ is a subsequence in $A \cap \mathbb{R}$ such that $\lim_{n\to\infty} b_n$ exists in $\mathbb{R} \cup \{\pm\infty\}$. Show that $\lim_{n\to\infty} b_n$ belongs to A.

Exercise 8. Let $\{a_n\}_{n\geq 1}$ be a sequence of non-negative numbers. For $n \geq 1$, define

$$s_n = \frac{a_1 + \ldots + a_n}{n}.$$

(i) Show that

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} a_n.$$

(ii) Conclude that if $\lim_{n\to\infty} a_n$ exists, then $\lim_{n\to\infty} s_n$ exists and $\lim_{n\to\infty} s_n = \lim_{n\to\infty} a_n$.

Exercise 9. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers that is bounded above. Prove that $L = \limsup a_n$ has the following properties:

(i) For every $\varepsilon > 0$ there are only finitely many n for which $a_n > L + \varepsilon$

(ii) For every $\varepsilon > 0$ there are infinitely many n for which $a_n > L - \varepsilon$.

Exercise 10. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Prove that there can be at most one real number L with the following two properties:

(i) For every $\varepsilon > 0$ there are only finitely many n for which $a_n > L + \varepsilon$

(ii) For every $\varepsilon > 0$ there are infinitely many n for which $a_n > L - \varepsilon$.