HOMEWORK 4

Exercise 1. Show that if a sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers converges to a, then the sequence $\{|a_n|\}_{n\in\mathbb{N}}$ converges to $|a|$. Show (via an example) that the converse is not true.

Exercise 2. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of rational numbers defined as follows:

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = a_n + \frac{1}{3^n} \quad \text{for all} \quad n \geq 1.$$

Show that the sequence $\{a_n\}_{n\in\mathbb{N}}$ converges and find its limit.

Exercise 3. Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two sequences of real numbers such that $\{a_n\}_{n\geq 1}$ is bounded and $\{b_n\}_{n\geq 1}$ converges to 0. Show that the sequence $\{a_nb_n\}_{n\geq 1}$ converges to 0.

Exercise 4. Let $\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$ and $\{c_n\}_{n\geq 1}$ be three convergent sequences of real numbers such that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n \quad \text{and} \quad a_n \leq b_n \leq c_n \text{ for all } n \geq 1.$$

Show that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$.

Exercise 5. Prove that

$$\lim_{n\to\infty} \sqrt{4n^2+n} - 2n = \tfrac{1}{4}.$$

Exercise 6. Let $\{a_n\}_{n\geq 1}$ be a convergent sequence of real numbers.
1) Show that if for all but finitely many a_n we have $a_n \geq a$, then $\lim_{n\to\infty} a_n \geq a$.
2) Show that if for all but finitely many a_n we have $a_n \leq b$, then $\lim_{n\to\infty} a_n \leq b$.
3) Conclude that if all but finitely many a_n belong to the interval $[a, b]$, then $\lim_{n\to\infty} a_n \in [a, b]$.

Exercise 7. Let $\{a_n\}_{n\geq 1}$ be a convergent sequence of real numbers and let $a \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n > a$. Show that there exist $n_0 \in \mathbb{N}$ such that $a_n > a$ for all $n \geq n_0$.

Exercise 8. Let $\{a_n\}_{n\geq 1}$ be a Cauchy sequence of real numbers. Show that $\{a_n^2\}_{n\geq 1}$ is also a Cauchy sequence.

Exercise 9. (In this exercise you will see a Cauchy sequence of rational numbers converging to an irrational number.) Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence defined by the following rule:

$$a_1 = 3 \quad \text{and} \quad a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n} \quad \text{for all} \quad n \geq 1.$$

1) Show that the sequence is bounded below.
2) Show that this is a sequence of rational numbers.
3) Prove that the sequence is monotonically decreasing.
4) Deduce that $\{a_n\}_{n\in\mathbb{N}}$ converges and find its limit.

Exercise 10. Consider the following sequence:

$$a_1 = \sqrt{2} \quad \text{and} \quad a_{n+1} = \sqrt{2 + a_n} \quad \text{for all } n \geq 1.$$

1) Show that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is bounded above.
2) Prove that the sequence is monotonically increasing.
3) Deduce that $\{a_n\}_{n\in\mathbb{N}}$ converges and find its limit.

Exercise 11. Let a_1, b_1 be two real numbers such that $0 < a_1 < b_1$. For $n \geq 1$, we define

$$a_{n+1} = \sqrt{a_n b_n} \quad \text{and} \quad b_{n+1} = \frac{a_n + b_n}{2}.$$

1) Prove that the sequence $\{a_n\}_{n\in N}$ is monotonically increasing and that the sequence $\{b_n\}_{n\in N}$ is monotonically decreasing.
2) Show that the sequences $\{a_n\}_{n\in N}$ and $\{b_n\}_{n\in N}$ are bounded.
3) Deduce that the two sequences converge and prove that they converge to the same limit.