HOMEWORK 3

Exercise 1. Let A and B be two non-empty bounded subsets of $\mathbb{R}$ and let

$$S = \{a + b : \; a \in A \text{ and } b \in B\}.$$

1) Prove that $\sup S = \sup A + \sup B$.

2) Prove that $\inf S = \inf A + \inf B$.

Exercise 2. Show that

$$\sup\{r \in \mathbb{Q} : \; r < a\} = a \quad \text{for all} \quad a \in \mathbb{R}.$$

Exercise 3. Let A and B be two non-empty bounded sets of *positive* real numbers and let

$$P = \{a \cdot b : \; a \in A \text{ and } b \in B\}.$$

Prove that $\sup P = \sup A \cdot \sup B$.

Exercise 4. Let $F_+ = \{\alpha : \alpha \text{ is a cut with } \alpha > \mathbf{0}\}$ be the set of positive Dedekind cuts, where we recall

$$\mathbf{0} = \{q \in \mathbb{Q} : \; q < 0\}.$$

We define the product of two elements $\alpha, \beta \in F_+$ via

$$\alpha \cdot \beta = \{r \in \mathbb{Q} : \; r < p \cdot q \text{ for some } 0 < p \in \alpha \text{ and } 0 < q \in \beta\}$$

Prove that this operation satisfies M1 through M5 on F_+.

Exercise 5. Let F be an ordered field with the least upper bound property and let $\phi : \mathbb{Q} \to F$ be the canonical injective function that satisfies the following properties:

$$\phi(p + q) = \phi(p) + \phi(q), \qquad \phi(p \cdot q) = \phi(p) \cdot \phi(q), \qquad \text{if } p < q \text{ then } \phi(p) < \phi(q)$$

for any $p, q \in \mathbb{Q}$.

For $x \in \mathbb{R}$, let

$$A_x = \{\phi(r) : \; r \in \mathbb{Q} \text{ with } r < x\}.$$

1) Show that A_x is a non-empty set bounded above. We define $\phi(x) = \sup A_x$.

2) Show that this extension of ϕ from $\mathbb{Q}$ to $\mathbb{R}$ satisfies the following: for any $x, y \in \mathbb{R}$,

$$\phi(x + y) = \phi(x) + \phi(y), \qquad \phi(x \cdot y) = \phi(x) \cdot \phi(y), \qquad \text{if } x < y \text{ then } \phi(x) < \phi(y).$$

3) Show that $\phi : \mathbb{R} \to F$ is bijective.