HOMEWORK 2

Exercise 1. Let $(F, +, \cdot, <)$ be an ordered field and let $a, b, c \in F$. Show that

$$2ab \leq a^2 + b^2$$

and

$$ab + bc + ca \leq a^2 + b^2 + c^2.$$

Specify what axioms you are using at each step.

Exercise 2. Let $(F, +, \cdot)$ be a field with exactly four distinct elements $F = \{0, 1, a, b\}$ where 0 and 1 denote the identities for $+$ and $\cdot$, respectively, and a, b denote the remaining two elements of F. Fill in the addition and multiplication tables below. Use the axioms to justify your answer. (Note that for each table entry there is a *unique* correct solution.)

$+$	0	1	a	b
0				
1				
a				
b				

$\cdot$	0	1	a	b
0				
1				
a				
b				

Hint: Show that in the addition table each row and each column contain every element of F exactly once (as in Sudoku). Show that the same is true for the rows and columns of the multiplication table that are not identically zero.

Exercise 3. Let $q \geq 2$ be a prime number. Recall the equivalence relation on $\mathbb{Z}$ defined as follows: for $m, n \in \mathbb{Z}$, we write $m \sim n$ if and only if $q|(m-n)$. For $n \in \mathbb{Z}$, denote by $C(n)$ the equivalence class of n. Let $\mathbb{Z}/q\mathbb{Z}$ denote the set of equivalence classes. We define addition and multiplication on $\mathbb{Z}/q\mathbb{Z}$ as follows:

$$C(n) + C(m) = C(n+m) \qquad \text{and} \qquad C(n) \cdot C(m) = C(nm).$$

1) Prove that addition and multiplication are well defined, that is, the result is independent of the representatives chosen from the equivalence classes.
2) Verify that with these operations $\mathbb{Z}/q\mathbb{Z}$ is a field.
3) Show that there is no order relation on $\mathbb{Z}/q\mathbb{Z}$ that makes it an ordered field.

Exercise 4. Define two internal laws of composition on $F = \mathbb{R} \times \mathbb{R}$ as follows:

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$
$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1).$$

1) Show that with these operations F is a field.
2) Show that there is no order relation on F that makes F an ordered field.

Definition 0.1. We say that a non-empty set R endowed with addition and multiplication is a ring if it satisfies the axioms $(A1), \ldots, (A5)$, $(M1), \ldots, (M4)$, and (D). If $<$ denotes an order relation on R satisfying the axioms (O1) and (O2), we say R is an ordered ring.

Exercise 5. Define two internal laws of composition on $R = \mathbb{Z} \times \mathbb{Z}$ as follows

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$
$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1 + 2a_2 b_2, a_1 b_2 + a_2 b_1).$$

1) Show that with these operations R is a ring.
2) For $(a_1, a_2), (b_1, b_2) \in R$ we write $(a_1, a_2) < (b_1, b_2)$ if $a_1 + a_2\sqrt{2} < b_1 + b_2\sqrt{2}$ in the usual sense on $\mathbb{R}$. Prove that this is an order relation on R and that with it, R is an ordered ring.

Exercise 6. Let S be a non-empty bounded subset of $\mathbb{R}$.
1) Prove that $\inf S \leq \sup S$.
2) What can you say about S if $\inf S = \sup S$?

Exercise 7. Let S and T be two non-empty bounded subsets of $\mathbb{R}$.
1) Prove that if $S \subseteq T$, then $\inf T \leq \inf S \leq \sup S \leq \sup T$.
2) Prove that $\sup(S \cup T) = \max\{\sup S, \sup T\}$.

Exercise 8. Let A be a non-empty subset of $\mathbb{R}$ which is bounded below and let

$$-A = \{-a : a \in A\}.$$

Prove that $\inf A = -\sup(-A)$.