## **HOMEWORK 2**

**Exercise 1.** Let  $(F, +, \cdot, <)$  be an ordered field and let  $a, b, c \in F$ . Show that

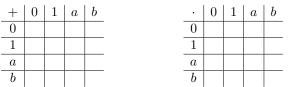
$$2ab \le a^2 + b^2$$

and

$$ab + bc + ca \le a^2 + b^2 + c^2.$$

Specify what axioms you are using at each step.

**Exercise 2.** Let  $(F, +, \cdot)$  be a field with exactly four distinct elements  $F = \{0, 1, a, b\}$  where 0 and 1 denote the identities for + and  $\cdot$ , respectively, and a, b denote the remaining two elements of F. Fill in the addition and multiplication tables below. Use the axioms to justify your answer. (Note that for each table entry there is a *unique* correct solution.)



*Hint:* Show that in the addition table each row and each column contain every element of F exactly once (as in Sudoku). Show that the same is true for the rows and columns of the multiplication table that are not identically zero.

**Exercise 3.** Let  $q \ge 2$  be a prime number. Recall the equivalence relation on  $\mathbb{Z}$  defined as follows: for  $m, n \in \mathbb{Z}$ , we write  $m \sim n$  if and only if q|(m-n). For  $n \in \mathbb{Z}$ , denote by C(n) the equivalence class of n. Let  $\mathbb{Z}/q\mathbb{Z}$  denote the set of equivalence classes. We define addition and multiplication on  $\mathbb{Z}/q\mathbb{Z}$  as follows:

$$C(n) + C(m) = C(n+m)$$
 and  $C(n) \cdot C(m) = C(nm)$ 

1) Prove that addition and multiplication are well defined, that is, the result is independent of the representatives chosen from the equivalence classes.

2) Verify that with these operations  $\mathbb{Z}/q\mathbb{Z}$  is a field.

3) Show that there is no order relation on  $\mathbb{Z}/q\mathbb{Z}$  that makes it an ordered field.

**Exercise 4.** Define two internal laws of composition on  $F = \mathbb{R} \times \mathbb{R}$  as follows:

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$
  
 $(a_1, a_2) \cdot (b_1, b_2) = (a_1b_1 - a_2b_2, a_1b_2 + a_2b_1).$ 

1) Show that with these operations F is a field.

2) Show that there is no order relation on F that makes F an ordered field.

**Definition 0.1.** We say that a non-empty set R endowed with addition and multiplication is a ring if it satisfies the axioms  $(A1), \ldots, (A5), (M1), \ldots, (M4)$ , and (D). If < denotes an order relation on R satisfying the axioms (O1) and (O2), we say R is an ordered ring.

**Exercise 5.** Define two internal laws of composition on  $R = \mathbb{Z} \times \mathbb{Z}$  as follows

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$
  
$$(a_1, a_2) \cdot (b_1, b_2) = (a_1b_1 + 2a_2b_2, a_1b_2 + a_2b_1).$$

1) Show that with these operations R is a ring.

2) For  $(a_1, a_2), (b_1, b_2) \in R$  we write  $(a_1, a_2) < (b_1, b_2)$  if  $a_1 + a_2\sqrt{2} < b_1 + b_2\sqrt{2}$  in the usual sense on  $\mathbb{R}$ . Prove that this is an order relation on R and that with it, R is an ordered ring.

**Exercise 6.** Let S be a non-empty bounded subset of  $\mathbb{R}$ .

1) Prove that  $\inf S \leq \sup S$ .

2) What can you say about S if  $\inf S = \sup S$ ?

**Exercise 7.** Let S and T be two non-empty bounded subsets of  $\mathbb{R}$ .

1) Prove that if  $S \subseteq T$ , then  $\inf T \leq \inf S \leq \sup S \leq \sup T$ .

2) Prove that  $\sup(S \cup T) = \max\{\sup S, \sup T\}.$ 

**Exercise 8.** Let A be a non-empty subset of  $\mathbb{R}$  which is bounded below and let

$$-A = \{-a : a \in A\}.$$

Prove that  $\inf A = -\sup(-A)$ .