HOMEWORK 9

Exercise 1. Let (X, d) be a metric space and let $\{x_n\}_{n\geq 1} \subseteq X$ be a convergent sequence. Prove that $\{x_n\}_{n\geq 1}$ is bounded, that is, there exist $a \in X$ and $r > 0$ such that $\{x_n\}_{n\geq 1} \subseteq B_r(a)$.

Exercise 2. Let $F \subseteq \mathbb{R}$ be a closed set bounded above. Prove that the least upper bound of F belongs to F.

Exercise 3. Let (X, d) be a metric space. Prove that a set $F \subseteq X$ is closed if and only if every convergent sequence in F has the property that its limit belongs to F.

Exercise 4. Let (X, d) be a metric space and let $A \subseteq X$ be complete. Show that A is closed.

Exercise 5. Let (X, d) be a complete metric space and let $F \subseteq X$ be a closed set. Show that F is complete.

Exercise 6. Let

$$l^\infty = \{\{x_n\}_{n\geq 1} \subseteq \mathbb{R} : \sup_{n\geq 1} |x_n| < \infty\}.$$

Define $d_\infty : l^\infty \times l^\infty \to \mathbb{R}$ as follows: for any $x = \{x_n\}_{n\geq 1} \in l^\infty$, $y = \{y_n\}_{n\geq 1} \in l^\infty$,

$$d_\infty(x, y) = \sup_{n\geq 1} |x_n - y_n|.$$

Show that (l^∞, d_∞) is a complete metric space.

Exercise 7. Let $\mathbb{R}^n$ be endowed with the Euclidean metric d_2. Let S be a non-empty subset of $\mathbb{R}^n$; in particular, $(S, d_2|_{S\times S})$ is a metric space.
1) Given $x \in S$, is the set $\{y \in S : d_2(x, y) \geq r\}$ closed in S?
2) Given $x \in S$, is the set $\{y \in S : d_2(x, y) \geq r\}$ contained in the closure of $\{y \in S : d_2(x, y) > r\}$ in S?