HOMEWORK 5

Exercise 1. Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two bounded sequences. Show that

$$\limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Exercise 2. Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two bounded sequences of non-negative numbers. Show that

$$\limsup_{n\to\infty}(a_n b_n) \leq \big(\limsup_{n\to\infty} a_n\big)\big(\limsup_{n\to\infty} b_n\big).$$

Exercise 3. Show that a sequence $\{a_n\}_{n\geq 1}$ is bounded if and only if $\limsup |a_n| < \infty$.

Exercise 4. Let A denote the set of subsequential limits of a sequence $\{a_n\}_{n\geq 1}$. Suppose that $\{b_n\}_{n\geq 1}$ is a subsequence in $A \cap \mathbb{R}$ such that $\lim_{n\to\infty} b_n$ exists in $\mathbb{R} \cup \{\pm\infty\}$. Show that $\lim_{n\to\infty} b_n$ belongs to A.

Exercise 5. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers that is bounded above. Prove that $L = \limsup a_n$ has the following properties:

(i) For every $\varepsilon > 0$ there are only finitely many n for which $a_n > L + \varepsilon$

(ii) For every $\varepsilon > 0$ there are infinitely many n for which $a_n > L - \varepsilon$.

Exercise 6. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Prove that there can be at most one real number L with the following two properties:

(i) For every $\varepsilon > 0$ there are only finitely many n for which $a_n > L + \varepsilon$

(ii) For every $\varepsilon > 0$ there are infinitely many n for which $a_n > L - \varepsilon$.

Exercise 7. Let $\{a_n\}_{n\geq 1}$ be a sequence of non-negative numbers. For $n \geq 1$, define

$$s_n = \frac{a_1 + \ldots + a_n}{n}.$$

(i) Show that

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} a_n.$$

(ii) Conclude that if $\lim_{n\to\infty} a_n$ exists, then $\lim_{n\to\infty} s_n$ exists and $\lim_{n\to\infty} s_n = \lim_{n\to\infty} a_n$.

Exercise 8. Let $\{a_n\}_{n\geq 1}$ be a sequence such that $\liminf_{n\to\infty} |a_n| = 0$. Prove that there is a subsequence $\{a_{k_n}\}_{n\geq 1}$ such that the series $\sum_{n=1}^{\infty} a_{k_n}$ converges.

Exercise 9. Determine which of the following series converge. Justify your answers.

$$(1)\ \sum_{n\geq 1} \frac{n^4}{2^n} \qquad (2)\ \sum_{n\geq 1} \frac{2^n}{n!} \qquad (3)\ \sum_{n\geq 1} (-1)^n \qquad (4)\ \sum_{n\geq 0} \sin\big(\tfrac{n\pi}{3}\big)$$