HOMEWORK 4

Due on Friday, October 26th, in class.

Exercise 1. Let $\{a_n\}_{n\geq 1}$ be a Cauchy sequence of real numbers. Show that $\{a_n^2\}_{n\geq 1}$ is also a Cauchy sequence.

Exercise 2. (In this exercise you will see a Cauchy sequence of rational numbers converging to an irrational number.) Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence defined by the following rule:

$$a_1 = 3 \quad \text{and} \quad a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n} \quad \text{for all} \quad n \geq 1.$$

1) Show that the sequence is bounded below.
2) Show that this is a sequence of rational numbers.
3) Prove that the sequence is monotonically decreasing.
4) Deduce that $\{a_n\}_{n\in\mathbb{N}}$ converges and find its limit.

Exercise 3. Consider the following sequence:

$$a_1 = \sqrt{2} \quad \text{and} \quad a_{n+1} = \sqrt{2 + a_n} \quad \text{for all } n \geq 1.$$

1) Show that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is bounded above.
2) Prove that the sequence is monotonically increasing.
3) Deduce that $\{a_n\}_{n\in\mathbb{N}}$ converges and find its limit.

Exercise 4. Let a_1, b_1 be two real numbers such that $0 < a_1 < b_1$. For $n \geq 1$, we define

$$a_{n+1} = \sqrt{a_n b_n} \quad \text{and} \quad b_{n+1} = \frac{a_n + b_n}{2}.$$

1) Prove that the sequence $\{a_n\}_{n\in N}$ is monotonically increasing and that the sequence $\{b_n\}_{n\in N}$ is monotonically decreasing.
2) Show that the sequences $\{a_n\}_{n\in N}$ and $\{b_n\}_{n\in N}$ are bounded.
3) Deduce that the two sequences converge and prove that they converge to the same limit.

Exercise 5. Let $\alpha > 1$ and define the sequence $\{x_n\}_{n\geq 1}$ of real numbers as follows:

$$x_1 > \sqrt{\alpha} \quad \text{and} \quad x_{n+1} = \frac{x_n + \alpha}{x_n + 1} \quad \text{for all} \quad n \geq 1.$$

1) Show that $\{x_{2n-1}\}_{n\geq 1}$ is decreasing and bounded below by $\sqrt{\alpha}$.
2) Show that $\{x_{2n}\}_{n\geq 1}$ is increasing and bounded above by $\sqrt{\alpha}$.
3) Show that the sequence $\{x_n\}_{n\geq 1}$ converges to $\sqrt{\alpha}$.

Exercise 6. Let

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = \left[1 - \tfrac{1}{(n+1)^2}\right] a_n \text{ for all } n \geq 1.$$

1) Show that the sequence $\{a_n\}_{n\geq 1}$ converges.
2) Find its limit.

Exercise 7. Let A be a non-empty bounded subset of $\mathbb{R}$ and suppose $\sup A \notin A$. Show that there exists an increasing sequence of points $\{a_n\}_{n\geq 1}$ in A such that $\lim_{n\to\infty} a_n = \sup A$.

Exercise 8. Let $\mathcal{C}$ be the set of Cauchy sequences of rational numbers. Define the relation $\sim$ as follows: if $\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1} \in \mathcal{C}$, we write $\{a_n\}_{n\geq 1} \sim \{b_n\}_{n\geq 1}$ if the sequence $\{a_n - b_n\}_{n\geq 1}$ converges to zero.
1) Prove that $\sim$ is an equivalence relation on $\mathcal{C}$.
2) For $\{a_n\}_{n\geq 1} \in \mathcal{C}$, we denote its equivalence class by $[a_n]$. Let R denote the set of equivalence classes in $\mathcal{C}$. We define addition and multiplication on R as follows:

$$[a_n] + [b_n] = [a_n + b_n] \quad \text{and} \quad [a_n] \cdot [b_n] = [a_n b_n].$$

Show that these internal laws of composition are well defined and that R together with these operations is a field.
3) We define a relation on R as follows: we write $[a_n] < [b_n]$ if $[a_n] \neq [b_n]$ and there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $a_n < b_n$. Prove that this relation is well defined. Show that the set of positive elements in R, that is,

$$P = \{[a_n] \in R : [a_n] > 0\}$$

satisfies the following properties:
01') For every $[a_n] \in R$, exactly one of the following holds: either $[a_n] = [0]$ or $[a_n] \in P$ or $-[a_n] \in P$, where $[0]$ denotes the equivalence class of the sequence identically equal to zero.
02') For every $[a_n], [b_n] \in P$, we have $[a_n] + [b_n] \in P$ and $[a_n] \cdot [b_n] \in P$.
Conclude that R is an ordered field.

Warning: You may not use that Cauchy sequences converge in $\mathbb{R}$. The purpose of the exercise is to give another construction of an ordered field with the least upper bound property.