HOMEWORK 3

Due on Friday, October 19th, in class.

Exercise 1. Let $q \geq 2$ be a prime number. Recall the equivalence relation on $\mathbb{Z}$ defined as follows: for $m, n \in \mathbb{Z}$, we write $m \sim n$ if $q|(m-n)$. For $n \in \mathbb{Z}$, denote by $C(n)$ the equivalence class of n. Let $\mathbb{Z}/q\mathbb{Z}$ denote the set of equivalence classes. Define addition and multiplication on $\mathbb{Z}/q\mathbb{Z}$ as follows:

$$C(n) + C(m) = C(n+m) \qquad \text{and} \qquad C(n) \cdot C(m) = C(nm).$$

1) Prove that addition and multiplication are well defined, that is, the result is independent of the representatives chosen from the equivalence classes.
2) Verify that with these operations $\mathbb{Z}/q\mathbb{Z}$ is a field.
3) Show that there is no order relation on $\mathbb{Z}/q\mathbb{Z}$ that makes it an ordered field.

Exercise 2. Define two internal laws of composition on $R = \mathbb{R} \times \mathbb{R}$ as follows:

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$
$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1).$$

1) Show that with these operations R is a field.
2) Show that there is no order relation on R that makes R an ordered ring.

Exercise 3. Show that if a sequence $\{a_n\}_{n \in \mathbb{N}}$ of real numbers converges to a, then the sequence $\{|a_n|\}_{n \in \mathbb{N}}$ converges to $|a|$. Show (via an example) that the converse is not true.

Exercise 4. Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of rational numbers defined as follows:

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = a_n + \frac{1}{3^n} \quad \text{for all} \quad n \geq 1.$$

Show that the sequence $\{a_n\}_{n \in \mathbb{N}}$ converges and find its limit.

Exercise 5. Let $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$ be two sequences of real numbers such that $\{a_n\}_{n \geq 1}$ is bounded and $\{b_n\}_{n \geq 1}$ converges to 0. Show that the sequence $\{a_n b_n\}_{n \geq 1}$ converges to 0.

Exercise 6. Let $\{a_n\}_{n \geq 1}$, $\{b_n\}_{n \geq 1}$ and $\{c_n\}_{n \geq 1}$ be three convergent sequences of real numbers such that

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n \quad \text{and} \quad a_n \leq b_n \leq c_n \text{ for all } n \geq 1.$$

Show that $\lim_{n \to \infty} b_n = \lim_{n \to \infty} a_n$.

Exercise 7. Prove that

$$\lim_{n \to \infty} \sqrt{4n^2 + n} - 2n = \tfrac{1}{4}.$$

Exercise 8. Let $\{a_n\}_{n \geq 1}$ be a convergent sequence of real numbers.
1) Show that if for all but finitely many a_n we have $a_n \geq a$, then $\lim_{n \to \infty} a_n \geq a$.
2) Show that if for all but finitely many a_n we have $a_n \leq b$, then $\lim_{n \to \infty} a_n \leq b$.
3) Conclude that if all but finitely many a_n belong to the interval $[a, b]$, then $\lim_{n \to \infty} a_n \in [a, b]$.

Exercise 9. Let $\{a_n\}_{n\geq 1}$ be a convergent sequence of real numbers and let $a \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n > a$. Show that there exist $n_0 \in \mathbb{N}$ such that $a_n > a$ for all $n \geq n_0$.