

HOMEWORK 2

Due on Friday, October 12, in class.

Exercise 1. Let $(F, +, \cdot, <)$ be an ordered field and let $a, b, c \in F$. Show that

$$2ab \leq a^2 + b^2$$

and

$$ab + bc + ca \leq a^2 + b^2 + c^2.$$

Specify what axioms you are using at each step.

Exercise 2. Let $(F, +, \cdot)$ be a field with exactly four distinct elements $F = \{0, 1, a, b\}$ where 0 and 1 denote the identities for $+$ and \cdot , respectively, and a, b denote the remaining two elements of F . Fill in the addition and multiplication tables below. Use the axioms to justify your answer. (Note that for each table entry there is a *unique* correct solution.)

$+$	0	1	a	b
0				
1				
a				
b				

\cdot	0	1	a	b
0				
1				
a				
b				

Hint: Show that in the addition table each row and each column contain every element of F exactly once (as in Sudoku). Show that the same is true for the rows and columns of the multiplication table that are not identically zero.

Definition 0.1. We say that a non-empty set R endowed with addition and multiplication is a ring if it satisfies the axioms (A1), ..., (A5), (M1), ..., (M4), and (D). If $<$ denotes an order relation on R satisfying the axioms (O1) and (O2), we say R is an ordered ring.

Exercise 3. Define two internal laws of composition on $R = \mathbb{Z} \times \mathbb{Z}$ as follows

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$

$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1 + 2a_2 b_2, a_1 b_2 + a_2 b_1).$$

- 1) Show that with these operations R is a ring.
- 2) Define an order relation \leq on R as follows: we write $(a_1, a_2) \leq (b_1, b_2)$ if $a_1 + a_2\sqrt{2} \leq b_1 + b_2\sqrt{2}$ in the usual sense on \mathbb{R} . Prove that this is an order relation on R and that with it, R is an ordered ring.

Exercise 4. Let S be a non-empty bounded subset of \mathbb{R} .

- 1) Prove that $\inf S \leq \sup S$.
- 2) What can you say about S if $\inf S = \sup S$?

Exercise 5. Let S and T be two non-empty bounded subsets of \mathbb{R} .

- 1) Prove that if $S \subseteq T$, then $\inf T \leq \inf S \leq \sup S \leq \sup T$.
- 2) Prove that $\sup(S \cup T) = \max\{\sup S, \sup T\}$.

Exercise 6. Let A be a non-empty subset of \mathbb{R} which is bounded below and let

$$-A = \{-a : a \in A\}.$$

Prove that $\inf A = -\sup(-A)$.

Exercise 7. Let A and B be two non-empty bounded subsets of \mathbb{R} and let

$$S = \{a + b : a \in A \text{ and } b \in B\}.$$

1) Prove that $\sup S = \sup A + \sup B$.

2) Prove that $\inf S = \inf A + \inf B$.

Exercise 8. Show that

$$\sup\{r \in \mathbb{Q} : r < a\} = a \quad \text{for all } a \in \mathbb{R}.$$