HOMEWORK 10

Exercise 1. Consider the metric space $(X,d) = (\mathbb{R}, |\cdot|)$. For each of the following subsets of $\mathbb{R}$ decide if they are open, closed, or not open and not closed, connected or not connected. Also, in each case write down the set of accumulation points. Justify your answers.
1) $A = \mathbb{Q}$.
2) $A = \mathbb{Q} \cap [0,1]$.
3) $A = \{(-1)^n(1+\frac{1}{n})\}$.
4) $A = \bigcup_{n\in\mathbb{N}}[n, n+\frac{1}{n}]$.
5) $A = \bigcup_{n\in\mathbb{N}}[\frac{1}{2^{n+1}}, \frac{1}{2^n}]$.

Exercise 2. Give an example of a set $\emptyset \neq A \subsetneq \mathbb{Q}$ that that is both open and closed in $\mathbb{Q}$. Justify your answer.

Exercise 3. Assume that the sets A and B are separated and let $A_1 \subseteq A$ and $B_1 \subseteq B$. Prove that A_1 and B_1 are separated.

Exercise 4. Assume that the sets A and B are separated and that the sets A and C are separated. Prove that the sets A and $B \cup C$ are separated.

Exercise 5. If A and B are closed sets, prove that $A \setminus B$ and $B \setminus A$ are separated.

Exercise 6. Let (X,d) be a connected metric space and let A be a connected subset of X. Assume that the complement of A is the union of two separated sets B and C. Prove that $A\cup B$ and $A\cup C$ are connected. Prove also that if A is closed, then so are $A \cup B$ and $A \cup C$.

Exercise 7. Let (X,d) be a metric space and let A, B be two closed subsets of X such that $A \cup B$ and $A \cap B$ are connected. Prove that A is connected.

Exercise 8. Let $\{A_i\}_{i\in I}$ be a family of connected sets such that one set of the family intersects all the others. Prove that $\cup_{i\in I} A_i$ is connected.

Exercise 9. Let (X,d) be a metric space and let A, B be two subsets of X such that $A \cap B \neq \emptyset$ and $B \setminus A \neq \emptyset$. Assume also that B is connected. Prove that $Fr(A) \cap B \neq \emptyset$. Deduce that if X is connected, then every subset of X, other than $\emptyset$ or X, has at least one frontier point.

Exercise 10. Let (X,d) be a complete metric space and let $\{G_n\}_{n\geq 1}$ be a sequence of dense open subsets of X. Show that $\cap_{n\geq 1} G_n$ has the Baire property.

Exercise 11. Show that $\big(\mathbb{R} \setminus \mathbb{Q}\big) \cap [0,1]$ has the Baire property.