HOMEWORK 9

Exercise 1. Let (X, d) be a metric space. Prove that if a sequence $\{x_n\}_{n\geq 1} \subseteq X$ converges in X, then its limit is unique.

Exercise 2. Let (X, d) be a metric space. Prove that a sequence $\{x_n\}_{n\geq 1} \subseteq X$ converges to some $x \in X$ if and only if every subsequence of $\{x_n\}_{n\geq 1}$ converges to x.

Exercise 3. Let (X, d) be a metric space and let $\{x_n\}_{n\geq 1} \subseteq X$ be a convergent sequence. Prove that $\{x_n\}_{n\geq 1}$ is bounded, that is, there exist $a \in X$ and $r > 0$ such that $\{x_n\}_{n\geq 1} \subseteq B_r(a)$.

Exercise 4. Let (X, d) be a metric space and let $A \subseteq X$ be complete. Show that A is closed.

Exercise 5. Let (X, d) be a complete metric space and let $F \subseteq X$ be a closed set. Show that F is complete.

Exercise 6. Let

$$l^\infty = \{\{x_n\}_{n\geq 1} \subseteq \mathbb{R} | \sup_{n\geq 1} |x_n| < \infty\}.$$

Define $d_\infty : l^\infty \times l^\infty \to \mathbb{R}$ as follows: for any $x = \{x_n\}_{n\geq 1} \in l^\infty$, $y = \{y_n\}_{n\geq 1} \in l^\infty$,

$$d_\infty(x, y) = \sup_{n\geq 1} |x_n - y_n|.$$

Show that (l^∞, d_∞) is a complete metric space.

Exercise 7. Let $\mathbb{R}^n$ be endowed with the Euclidean metric d_2. Let S be a non-empty subset of $\mathbb{R}^n$; in particular, $(S, d_2|_{S\times S})$ is a metric space.
1) Given $x \in S$, is the set $\{y \in S : d_2(x, y) \geq r\}$ closed in S?
2) Given $x \in S$, is the set $\{y \in S : d_2(x, y) \geq r\}$ contained in the closure of $\{y \in S : d_2(x, y) > r\}$ in S?

Exercise 8. Let (X, d) be a complete metric space and let $\{F_n\}_{n\geq 1}$ be a sequence of non-empty closed subsets of X such that $F_{n+1} \subseteq F_n$ for all $n \geq 1$ and $\delta(F_n) \to 0$. Show that there exists $x \in X$ such that $\cap_{n\geq 1} F_n = \{x\}$.