HOMEWORK 6

Exercise 1. Study the convergence of the series

$$1) \sum_{n\geq 2} \frac{1}{[n+(-1)^n]^2} \qquad 2) \sum_{n\geq 1} [\sqrt{n+1}-\sqrt{n}] \qquad 3) \sum_{n\geq 1} \frac{n!}{n^n}$$

Exercise 2. Study the convergence of the series

$$1) \sum_{n\geq 2} \frac{n^{\ln n}}{(\ln n)^n} \qquad 2) \sum_{n\geq 2} \frac{1}{(\ln n)^{\ln n}} \qquad 3) \sum_{n\geq 1} \frac{(-1)^n n!}{2^n}.$$

Exercise 3.
(1) Give an example of a divergent series $\sum a_n$ for which $\sum a_n^2$ converges.
(2) Show that if $\sum a_n$ is absolutely convergent, then the series $\sum a_n^2$ also converges.
(3) Give an example of a convergent series $\sum a_n$, for which $\sum a_n^2$ diverges.

Exercise 4. Prove that

$$\sum_{n\geq 1} \frac{1}{n(n+1)} = 1.$$

Exercise 5.
(1) Prove that

$$\sum_{n\geq 1} \frac{n-1}{2^{n+1}} = \frac{1}{2}.$$

(2) Use part (1) to calculate

$$\sum_{n\geq 1} \frac{n}{2^n}.$$

Hint: Note that $\frac{n-1}{2^{n+1}} = \frac{n}{2^n} - \frac{n+1}{2^{n+1}}$.

Exercise 6. Let $\{a_n\}_{n\geq 1}$ be a sequence of non-negative numbers such that $\sum_{n\geq 1} a_n$ diverges. For $n \geq 1$, let $s_n = a_1 + \ldots + a_n$.
(1) Prove that the series

$$\sum_{n\geq 1} \frac{a_n}{a_n+1}$$

diverges.
(2) Prove that for all $N \geq 1$ and all $n \geq 1$,

$$\sum_{k=1}^{n} \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+n}}.$$

Deduce that the series $\sum \frac{a_n}{s_n}$ diverges.
(3) Prove that for all $n \geq 2$,

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}.$$

Deduce that the series $\sum \frac{a_n}{s_n^2}$ converges.

Exercise 7. Let $\{a_n\}_{n\geq 1}$ be a decreasing sequence of non-negative numbers such that $\sum_{n\geq 1} a_n < \infty$. Show that

$$\lim_{n\to\infty} na_n = 0.$$