HOMEWORK 2

Due on Friday, October 13th, in class.

Exercise 1. Let $(F,+,\cdot,<)$ be an ordered field and let $a,b,c\in F$. Show that

$$2ab \leq a^2+b^2$$

and

$$ab+bc+ca \leq a^2+b^2+c^2.$$

Specify what axioms you are using at each step.

Exercise 2. Let $(F,+,\cdot)$ be a field with exactly four distinct elements $F = \{0,1,a,b\}$ where 0 and 1 denote the identities for $+$ and $\cdot$, respectively, and a,b denote the remaining two elements of F. Fill in the addition and multiplication tables below. Use the axioms to justify your answer. (Note that for each table entry there is a *unique* correct solution.)

$+$	0	1	a	b
0				
1				
a				
b				

$\cdot$	0	1	a	b
0				
1				
a				
b				

Hint: Show that in the addition table each row and each column contain every element of F exactly once (as in Sudoku). Show that the same is true for the rows and columns of the multiplication table that are not identically zero.

Definition 0.1. We say that a non-empty set R endowed with addition and multiplication is a ring if it satisfies the axioms $(A1),\dots,(A5)$, $(M1),\dots,(M4)$, and (D). If $<$ denotes an order relation on R satisfying the axioms (O1) and (O2), we say R is an ordered ring.

Exercise 3. Define two internal laws of composition on $R=\mathbb{Z}\times\mathbb{Z}$ as follows

$$(a_1,a_2)+(b_1,b_2)=(a_1+b_1,a_2+b_2)$$
$$(a_1,a_2)\cdot(b_1,b_2)=(a_1b_1+2a_2b_2,a_1b_2+a_2b_1).$$

1) Show that with these operations R is a ring.
2) Define an order relation $\leq$ on R as follows: we write $(a_1,a_2)\leq(b_1,b_2)$ if $a_1+a_2\sqrt{2}\leq b_1+b_2\sqrt{2}$ in the usual sense on $\mathbb{R}$. Prove that this is an order relation on R and that with it, R is an ordered ring.

Exercise 4. Let S be a non-empty bounded subset of $\mathbb{R}$.
1) Prove that $\inf S\leq \sup S$.
2) What can you say about S if $\inf S=\sup S$?

Exercise 5. Let S and T be two non-empty bounded subsets of $\mathbb{R}$.
1) Prove that if $S\subseteq T$, then $\inf T\leq \inf S\leq \sup S\leq \sup T$.
2) Prove that $\sup(S\cup T)=\max\{\sup S,\sup T\}$.

Exercise 6. Let A be a non-empty subset of $\mathbb{R}$ which is bounded below and let

$$-A = \{-a : \, a \in A\}.$$

Prove that $\inf A = -\sup(-A)$.

Exercise 7. Let A and B be two non-empty bounded subsets of $\mathbb{R}$ and let

$$S = \{a + b : \, a \in A \text{ and } b \in B\}.$$

1) Prove that $\sup S = \sup A + \sup B$.
2) Prove that $\inf S = \inf A + \inf B$.

Exercise 8. Show that

$$\sup\{r \in \mathbb{Q} : \, r < a\} = a \quad \text{for all} \quad a \in \mathbb{R}.$$