

HOMEWORK 1

Due on Friday, October 6th, in class.

Exercise 1. Negate the following sentences:

- If pigs had wings, they would fly.
- If that plane leaves and you are not on it, then you will regret it.
- For every problem there is a solution that is neat, plausible, and wrong.

Exercise 2. Let X and Y be statements. If we want to DISPROVE the claim that “At least one of X and Y are true”, which one of the following do we need to show?

- a) At least one of X and Y are false.
- b) X and Y are both false.
- c) Exactly one of X and Y are false.
- d) Y is false.
- e) X does not imply Y , and Y does not imply X .
- f) X is true if and only if Y is false.
- g) X is false.

Exercise 3. Let $P(x)$ be a property about some object x of type X . If we want to DISPROVE the claim that “ $P(x)$ is true for all x of type X ”, which one of the following do we have to do?

- a) Show that for every x in X , $P(x)$ is false.
- b) Show that for every x in X , there is a y not equal to x for which $P(y)$ is true.
- c) Show that $P(x)$ being true does not necessarily imply that x is of type X .
- d) Show that there are no objects x of type X .
- e) Show that there exists an x of type X for which $P(x)$ is false.
- f) Show that there exists an x which is not of type X , but for which $P(x)$ is still true.
- g) Assume there exists an x of type X for which $P(x)$ is true, and derive a contradiction.

Exercise 4. Let X, Y, Z be statements. Suppose we know that “ X is true implies Y is true”, and “ X is false implies Z is true”. If we know that Z is false, then which one of the following can we conclude?

- a) X is false.
- b) X is true.
- c) Y is true.
- d) b) and c).
- e) a) and c).
- f) a), b), and c).
- g) None of the above conclusions can be drawn.

Exercise 5. Let $P(n, m)$ be a property about two integers n and m . If we want to DISPROVE the claim that “There exists an integer n such that $P(n, m)$ is true for all integers m ”, then which one of the following do we need to prove?

- a) If $P(n, m)$ is true, then n and m are not integers.
- b) For every integer n , there exists an integer m such that $P(n, m)$ is false.

- c) For every integer n , and every integer m , the property $P(n, m)$ is false.
- d) For every integer m , there exists an integer n such that $P(n, m)$ is false.
- e) There exists an integer n such that $P(n, m)$ is false for all integers m .
- f) There exists integers n, m such that $P(n, m)$ is false.
- g) There exists an integer m such that $P(n, m)$ is false for all integers n .

Exercise 6. Let X and Y be statements. If we know that X implies Y , which one of the following can we conclude?

- a) X cannot be false.
- b) X is true, and Y is also true.
- c) If Y is false, then X is false.
- d) Y cannot be false.
- e) If X is false, then Y is false.
- f) If Y is true, then X is true.
- g) At least one of X and Y is true.

Exercise 7. Prove the following statement by induction:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2 \quad \text{for all } n \geq 1.$$

Exercise 8. Prove the following statement by induction: The expression $4^n + 15n - 1$ is divisible by 9 for all $n \geq 1$.

Exercise 9. Decide for which natural numbers the inequality $3^n > n^3$ is true. Prove your claim using mathematical induction.

Exercise 10. Prove there is no rational number whose square is 6.