OBERWOLFACH SEMINAR: DISPERSIVE EQUATIONS

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Contents

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While the principal result presented here is from the paper [40], the exposition owes much to a long series of joint works with my collaborators Rowan Killip and Xiaoyi Zhang. We also reproduce here some of the innovations and simplifications introduced in [22]; those notes also contain a far more detailed discussion of the history. Finally, I am grateful to Casey Jao and Jason Murphy for their careful reading of these notes.

1. Notation

Throughout this text, we will be regularly referring to the spacetime norms

$$\|u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} := \left(\int_{\mathbb{R}} \left[\int_{\mathbb{R}^d} |u(t,x)|^r \, dx\right]^{\frac{q}{r}} dt\right)^{\frac{1}{q}}, \tag{1.1}$$

with obvious changes if q or r are infinity. We will often use the abbreviation

$$\|f\|_r := \|f\|_{L_x^r} \quad \text{and} \quad \|u\|_{q,r} := \|u\|_{L_t^q L_x^r}.$$

We write $X \lesssim Y$ to indicate that $X \leq CY$ for some constant C, which is permitted to depend on the ambient spatial dimension, d, without further comment. Other dependencies of C will be indicated with subscripts, for example, $X \lesssim_u Y$. We will write $X \sim Y$ to indicate that $X \lesssim Y \lesssim X$.

We use the 'Japanese bracket' convention: $\langle x \rangle := (1+|x|^2)^{1/2}$ as well as $\langle \nabla \rangle := (1-\Delta)^{1/2}$. Similarly, $|\nabla|^s$ denotes the Fourier multiplier with symbol $|\xi|^s$. These are used to define the Sobolev norms

$$\|f\|_{H^{s,r}} := \|\langle \nabla \rangle^s f\|_{L^r_x} \quad \text{and} \quad \|f\|_{\dot{H}^{s,r}} := \||\nabla|^s f\|_{L^r_x}.$$

When $r=2$ we abbreviate $H^s = H^{s,2}$ and $\dot{H}^s = \dot{H}^{s,2}$.

Our convention for the Fourier transform is

$$\hat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx$$

so that

$$f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi \quad \text{and} \quad \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2\, d\xi = \int_{\mathbb{R}^d} |f(x)|^2\, dx.$$

Notations associated to Littlewood–Paley projections are discussed in Appendix A.

2. Dispersive and Strichartz estimates

What all linear dispersive-type equations have in common is a dispersive-type estimate, which expresses the fact that wave-packets spread out as time goes by. An expression of this on the Fourier side is the fact that different frequencies move with different speeds and/or in different directions. Below we will discuss several instances of this phenomenon.

2.1. **The linear Schrödinger equation.**

The initial-value problem for the linear Schrödinger equation takes the form

$$i\partial_t u = -\Delta u \quad \text{with} \quad u(0,x) = u_0(x). \tag{2.1}$$

Here u denotes a complex-valued function of spacetime $\mathbb{R}_t \times \mathbb{R}^d_x$ with the spatial dimension $d \geq 1$. By taking Fourier transforms, we observe that

$$\hat{u}(t,\xi) = e^{-it|\xi|^2} \hat{u}_0(\xi). \tag{2.2}$$

In particular, solutions with Schwartz initial data are Schwartz for all $t \in \mathbb{R}$.

Using (2.2) and Plancherel, it is easy to see that solutions to (2.1) conserve *mass*, that is,

$$\|e^{it\Delta} u_0\|^2_{L^2_x} = \|u_0\|^2_{L^2_x}, \tag{2.3}$$

and *kinetic energy*, that is,

$$\|\nabla e^{it\Delta} u_0\|^2_{L^2_x} = \|\nabla u_0\|^2_{L^2_x}.$$

To derive an explicit formula for solutions to (2.1), we will first study the particular case of modulated Gaussian initial data, namely,

$$u_0(x) = \exp\bigl\{-\tfrac{|x|^2}{4\sigma^2} + ix\xi_0\bigr\} \quad \text{with} \quad \sigma > 0 \quad \text{and} \quad \xi_0 \in \mathbb{R}^d.$$

This initial data is a Gaussian that lives at scale σ and has wave vector ξ_0, that is, it has wave length $\frac{2\pi}{|\xi_0|}$ and the wave fronts are perpendicular to ξ_0. A straightforward computation yields that the solution u to (2.1) with this initial data is given by

$$\begin{aligned}
[e^{it\Delta}u_0](x) &= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{ix\xi - it|\xi|^2} \hat{u}_0(\xi)\, d\xi \\
&= (2\pi)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ix\xi - it|\xi|^2} e^{-iy\xi} e^{-\frac{|y|^2}{4\sigma^2} + iy\xi_0}\, dy\, d\xi \\
&= (2\pi)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ix\xi - it|\xi|^2} e^{-\sigma^2|\xi-\xi_0|^2} e^{-|\frac{y}{2\sigma} + i\sigma(\xi-\xi_0)|^2}\, dy\, d\xi \\
&= (2\pi)^{-d}(4\pi\sigma^2)^{\frac{d}{2}} e^{-it|\xi_0|^2 + ix\xi_0 - \frac{|x-2t\xi_0|^2}{4(\sigma^2+it)}} \int_{\mathbb{R}^d} e^{-(\sigma^2+it)\left|\xi - \frac{ix+2\sigma^2\xi_0}{2(\sigma^2+it)}\right|^2}\, d\xi \\
&= \Big(\frac{\sigma^2}{\sigma^2+it}\Big)^{\frac{d}{2}} \exp\big\{-it|\xi_0|^2 + ix\xi_0 - \tfrac{|x-2t\xi_0|^2}{4(\sigma^2+it)}\big\}. \qquad (2.4)
\end{aligned}$$

In the formulas above, $|v|^2 := \sum_{j=1}^d v_j^2$ for all vectors $v \in \mathbb{C}^d$.

Exercise 2.1. Justify all steps in the derivation of (2.4).

Remark. From the exact formula (2.4), we read the following:
- the wave-packet travels at speed $2\xi_0$ (called the *group velocity*)
- the wave vector is still ξ_0 (called the *phase velocity*)
- while the amplitude of the wave packet decreases with time, the wave-packet also spreads out: Re $\frac{1}{4(\sigma^2+it)} < \frac{1}{4\sigma^2}$. This is consistent with the conservation of mass.

We are now ready to derive an exact formula for solutions to (2.1), at least for Schwartz initial data $u_0 \in \mathcal{S}(\mathbb{R}^d)$. Using the linearity of the propagator $e^{it\Delta}$ and (2.4), we get

$$e^{it\Delta}\Big[(4\pi\sigma^2)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4\sigma^2}} u_0(y)\, dy\Big] = [4\pi(\sigma^2+it)]^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4(\sigma^2+it)}} u_0(y)\, dy.$$

To continue, the key observation is that for $u_0 \in \mathcal{S}(\mathbb{R}^d)$,

$$\lim_{\sigma\to 0} (4\pi\sigma^2)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4\sigma^2}} u_0(y)\, dy = u_0(x) \qquad (2.5)$$

both pointwise in x and in the L^2_x topology. Using also that the propagator $e^{it\Delta}$ is continuous in the L^2_x topology (on Schwartz space), we get the exact formula

$$[e^{it\Delta}u_0](x) = (4\pi it)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t}} u_0(y)\, dy \quad \text{for} \quad t \neq 0 \qquad (2.6)$$

for all $u_0 \in \mathcal{S}(\mathbb{R}^d)$, where the equality is meant in the L^2_x sense.

This leads directly to the dispersive inequality for the linear Schrödinger propagator:

$$\|e^{it\Delta}u_0\|_{L^\infty_x} \lesssim |t|^{-\frac{d}{2}} \|u_0\|_{L^1_x} \quad \text{for} \quad t \neq 0. \qquad (2.7)$$

Interpolating with (2.3), we obtain the full range of dispersive estimates for the linear Schrödinger propagator:

$$\|e^{it\Delta}u_0\|_{L^p_x} \lesssim |t|^{\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})} \|u_0\|_{L^{p'}_x} \quad \text{for} \quad t \neq 0 \qquad (2.8)$$

for all $2 \leq p \leq \infty$, where p' denotes the exponent conjugate to p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

Exercise 2.2. Prove that for all $u_0 \in L^2_x$, the equality (2.5) holds both a.e. in x and in the L^2_x topology.

2.2. **The Airy equation.** The initial-value problem for the Airy equation takes the form

$$\partial_t u = -\partial_x^3 u \quad \text{with} \quad u(0,x) = u_0(x). \tag{2.9}$$

Here u denotes a real-valued function of spacetime $\mathbb{R}_t \times \mathbb{R}_x$. Note that complex-valued solutions to (2.9) have the property that their real and imaginary parts individually solve (2.9).

Using the Fourier transform, we arrive at

$$[e^{-t\partial_x^3} u_0](x) = (3t)^{-1/3} \int_{\mathbb{R}} \mathrm{Ai}\big(\tfrac{x-y}{(3t)^{1/3}}\big) u_0(y)\, dy \quad \text{for} \quad t \neq 0, \tag{2.10}$$

where

$$\mathrm{Ai}(x) := \pi^{-1} \int_0^\infty \cos(\tfrac{1}{3}\xi^3 + x\xi)\, d\xi$$

denotes the Airy function of the first kind.

Exercise 2.3. Prove that the Airy function is uniformly bounded. Indeed, show that $\mathrm{Ai}(x) \to 0 \quad \text{as} \quad x \to \pm\infty$.
Hint: Use non-stationary phase when $x \geq 1$; van der Corput for $|x| \leq 1$; van der Corput for $x \leq -1$ on $|\xi| \sim |x|^{1/2}$ and the complementary region, separately.

As a consequence of this exercise and (2.10), we obtain the dispersive estimate for the Airy equation:

$$\|e^{-t\partial_x^3} u_0\|_{L^\infty_x} \lesssim |t|^{-\frac{1}{3}} \|u_0\|_{L^1_x} \quad \text{for} \quad t \neq 0. \tag{2.11}$$

Interpolating with the conservation law

$$\|e^{-t\partial_x^3} u_0\|_{L^2_x} = \|u_0\|_{L^2_x},$$

we obtain the full range of dispersive estimates, namely,

$$\|e^{-t\partial_x^3} u_0\|_{L^p_x} \lesssim |t|^{\frac{1}{3}(\frac{1}{p} - \frac{1}{p'})} \|u_0\|_{L^{p'}_x} \quad \text{for} \quad t \neq 0 \tag{2.12}$$

for all $2 \leq p \leq \infty$, where p' denotes the exponent conjugate to p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

We may strengthen the dispersive estimate (2.11) by localizing in frequency:

Exercise 2.4 (Frequency-localized dispersive estimate for the Airy propagator)**.** Let $f \in \mathcal{S}(\mathbb{R})$. Prove that

$$\|e^{-t\partial_x^3} P_N u_0\|_{L^\infty_x} \lesssim \min\{|t|^{-\frac{1}{3}}, (N|t|)^{-\frac{1}{2}}\} \|P_N u_0\|_{L^1_x}$$

uniformly for $N \in 2^{\mathbb{Z}}$ and $t \neq 0$. Here P_N denotes the Littlewood–Paley projection to frequencies $|\xi| \sim N$; see Appendix (A) for definitions and basic properties.

2.3. **The linear wave equation.** The initial-value problem for the linear wave equation takes the form

$$\partial_t^2 u = \Delta u \quad \text{with} \quad u(0,x) = u_0(x) \quad \text{and} \quad \partial_t u(0,x) = u_1(x). \tag{2.13}$$

Here u denotes a real-valued function of spacetime $\mathbb{R}_t \times \mathbb{R}^d_x$ with the spatial dimension $d \geq 1$.

Using the Fourier transform, we find

$$\begin{pmatrix} u \\ u_t \end{pmatrix}(t) = \begin{pmatrix} \cos(t|\nabla|) & |\nabla|^{-1}\sin(t|\nabla|) \\ -|\nabla|\sin(t|\nabla|) & \cos(t|\nabla|) \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

One can derive an explicit formula for the wave propagator in spatial variables; see, for example, [31]. One advantage of this expression is that it immediately yields Huygens' principle. This exact formula can also be used to derive the dispersive estimate we give below; however, we prefer to take a Fourier analytic approach that generalizes to more equations.

Lemma 2.1 (Frequency-localized dispersive estimate for the half-wave propagator)**.** *For any $d \geq 1$ and any frequency $N \in 2^{\mathbb{Z}}$, we have*

$$\|e^{\pm it|\nabla|}P_N f\|_{L^\infty_x} \lesssim \big(1+|t|N\big)^{-\frac{d-1}{2}} N^d \|P_N f\|_{L^1_x}. \tag{2.14}$$

In particular, interpolating with $\|e^{it|\nabla|}P_N f\|_{L^2_x} = \|P_N f\|_{L^2_x}$ we get

$$\|e^{\pm it|\nabla|}P_N f\|_{L^p_x} \lesssim \big(1+|t|N\big)^{-\frac{(d-1)(p-2)}{2p}} N^{\frac{d(p-2)}{p}} \|P_N f\|_{L^{p'}_x}, \tag{2.15}$$

for all $2 \leq p \leq \infty$, where p' denotes the exponent conjugate to p, that is, $\frac{1}{p}+\frac{1}{p'}=1$.

Proof. By symmetry, it suffices to prove the dispersive estimate for the propagator $e^{it|\nabla|}$. If $d=1$ or $d \geq 2$ and $|t| \lesssim N^{-1}$, the claim (2.14) follows easily from the Bernstein inequality:

$$\|e^{it|\nabla|}P_N f\|_{L^\infty_x} \lesssim N^{\frac{d}{2}}\|e^{it|\nabla|}P_N f\|_{L^2_x} \lesssim N^{\frac{d}{2}}\|P_N f\|_{L^2_x} \lesssim N^d \|P_N f\|_{L^1_x}.$$

It thus remains to prove the claim for $d \geq 2$ and $|t| \gg N^{-1}$, to which we now turn. We write

$$e^{it|\nabla|}P_N f = e^{it|\nabla|}\tilde P_N f_N = \big[e^{it|\xi|}\tilde\psi\big(\tfrac{\xi}{N}\big)\widehat{f_N}(\xi)\big]^\vee = \big[e^{it|\xi|}\tilde\psi\big(\tfrac{\xi}{N}\big)\big]^\vee * f_N,$$

where $\tilde P_N = P_{N/2}+P_N+P_{2N}$ denotes the fattened Littlewood–Paley projection, ψ denotes the Fourier multiplier associated with P_1, and $\tilde\psi$ denotes the Fourier multiplier associated with $\tilde P_1$. To establish (2.14), it thus suffices to show

$$\Big|\int_{\mathbb{R}^d} e^{ix\xi+it|\xi|}\tilde\psi\big(\tfrac{\xi}{N}\big)\,d\xi\Big| \lesssim N^{\frac{d+1}{2}}|t|^{-\frac{d-1}{2}} \tag{2.16}$$

for all $d \geq 2$ and $|t| \gg N^{-1}$.

Using a change of variables and switching to polar coordinates, we write

$$\int_{\mathbb{R}^d} e^{ix\xi+it|\xi|}\tilde\psi\big(\tfrac{\xi}{N}\big)\,d\xi = N^d\int_0^\infty\int_{S^{d-1}} e^{ixNr\omega+itNr}\tilde\psi(r)\,d\sigma(\omega)r^{d-1}\,dr \tag{2.17}$$

$$= N^d\int_0^\infty e^{itNr}\tilde\psi(r)\check\sigma(Nr|x|)r^{d-1}\,dr, \tag{2.18}$$

where $d\sigma$ denotes the surface measure on the sphere $S^{d-1}\subset\mathbb{R}^d$.

If $|x| \ll |t|$, we note that the phase $\phi(r) := Nrx\omega + Nrt$ has no critical points; indeed, $|\phi'(r)| \gtrsim N|t|$. Thus, writing $e^{i\phi(r)} = \frac{1}{i\phi'(r)}\partial_r e^{i\phi(r)}$ and integrating by parts k times in (2.17), we get the bound

$$\Big|\int_{\mathbb{R}^d} e^{ix\xi+it|\xi|}\tilde\psi\big(\tfrac{\xi}{N}\big)\,d\xi\Big| \lesssim_k N^d(N|t|)^{-k} \lesssim N^{\frac{d+1}{2}}|t|^{-\frac{d-1}{2}}.$$

To obtain the last inequality, we take $k = \frac{d-1}{2}$ if the dimension d is odd, or $k = \frac{d}{2}$ if the dimension d is even (recalling that $|t| \gg N^{-1}$).

It remains to consider the case $|x| \gtrsim |t|$. In this case we use (2.18) together with the following lemma:

Lemma 2.2. *Let $d \geq 2$ and let $d\sigma$ denote the surface measure on the sphere $S^{d-1} \subset \mathbb{R}^d$. Then*

$$|\check{\sigma}(x)| \lesssim \langle x\rangle^{-\frac{d-1}{2}}.$$

Proof. Exercise! *Hint:* Using the fact that $d\sigma$ is rotationally invariant, we may write

$$\check{\sigma}(x) = (2\pi)^{-\frac{d}{2}} \int_{S^{d-1}} e^{i|x|\xi_d} d\sigma(\xi) \sim \int_0^\pi e^{i|x|\cos\theta}(\sin\theta)^{d-2}\, d\theta,$$

where θ is the angle x makes with e_d. Now use stationary phase and van der Corput. □

Returning to the proof of Lemma 2.1, for $|x| \gtrsim |t| \gg N^{-1}$ we use (2.18) and Lemma 2.2 to estimate

$$\left|\int_{\mathbb{R}^d} e^{ix\xi+it|\xi|}\tilde{\psi}\big(\tfrac{\xi}{N}\big)\, d\xi\right| \lesssim N^d \int_{\mathbb{R}^d} |\tilde{\psi}(r)|(Nr|x|)^{-\frac{d-1}{2}} r^{d-1}\, dr \lesssim N^{\frac{d+1}{2}}|t|^{-\frac{d-1}{2}},$$

which gives (2.16) in this case. This completes the proof of (2.16) and so the proof of Lemma 2.14. □

2.4. **From dispersive to Strichartz estimates.** In this subsection, we will only present details for the derivation of Strichartz estimates for the wave equation. Strichartz estimates for Schrödinger and Airy are left as exercises for the reader.

Definition 2.3. We say that (q, r) is *wave admissible* if

$$\frac{1}{q} + \frac{d-1}{2r} \leq \frac{d-1}{4}, \quad q, r, d \geq 2, \quad \text{and} \quad (q, r, d) \neq (2, \infty, 3).$$

Proposition 2.4 (Frequency-localized Strichartz estimates for the half-wave propagator)**.** *Let $d \geq 2$ and (q, r) be wave admissible such that $\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \gamma$ for some $\gamma > 0$. Then*

$$\|e^{\pm it|\nabla|}P_N f\|_{L_t^q L_x^r} \lesssim N^\gamma \|P_N f\|_{L_x^2} \tag{2.19}$$

$$\left\|\int_{\mathbb{R}} e^{\mp it|\nabla|}P_N F(t)\, dt\right\|_{L_x^2} \lesssim N^\gamma \|P_N F\|_{L_t^{q'} L_x^{r'}}. \tag{2.20}$$

Moreover, if $(\tilde{q}, \tilde{r})$ is also a wave admissible pair, then we have the retarded estimate

$$\left\|\int_{s<t} e^{\pm i(t-s)|\nabla|}P_N F(s)\, ds\right\|_{L_t^q L_x^r} \lesssim N^{d-\frac{1}{q}-\frac{1}{\tilde{q}}-\frac{d}{r}-\frac{d}{\tilde{r}}}\|P_N F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}. \tag{2.21}$$

Proof. We will only prove the proposition in the non-endpoint cases, that is, omitting the pair $(2, \frac{2(d-1)}{d-3})$ for $d > 3$. For the endpoint case, see [17].

By the TT^* argument, (2.19) is equivalent to (2.20) and they are both equivalent to

$$\left\|\int_{\mathbb{R}} e^{\pm i(t-s)|\nabla|}P_N F(s)\, ds\right\|_{L_t^q L_x^r} \lesssim N^{2\gamma}\|F\|_{L_t^{q'} L_x^{r'}}. \tag{2.22}$$

When $\frac{1}{q}+\frac{d-1}{2r}<\frac{d-1}{4}$ we use (2.15) and Young's inequality to estimate

$$\begin{aligned}\text{LHS(2.22)} &\lesssim \Big\| \int_{\mathbb{R}} \big(1+|t-s|N\big)^{-\frac{(d-1)(r-2)}{2r}} N^{\frac{d(r-2)}{r}} \|P_N F(s)\|_{L_x^{r'}}\, ds \Big\|_{L_t^q} \\ &\lesssim N^{\frac{d(r-2)}{r}} \|P_N F\|_{L_t^{q'} L_x^{r'}} \Big\| \big(1+|t|N\big)^{-\frac{(d-1)(r-2)}{2r}} \Big\|_{L_t^{q/2}} \\ &\lesssim N^{\frac{d(r-2)}{r}} N^{-\frac{2}{q}} \|P_N F\|_{L_t^{q'} L_x^{r'}},\end{aligned}$$

which gives (2.22) in this case. When $\frac{1}{q}+\frac{d-1}{2r}=\frac{d-1}{4}$ we use instead the Hardy–Littlewood–Sobolev inequality to obtain

$$\begin{aligned}\text{LHS(2.22)} &\lesssim \Big\| \int_{\mathbb{R}} |t-s|^{-\frac{(d-1)(r-2)}{2r}} N^{\frac{(d+1)(r-2)}{2r}} \|P_N F(s)\|_{L_x^{r'}}\, ds \Big\|_{L_t^q} \\ &\lesssim N^{\frac{(d+1)(r-2)}{2r}} \|P_N F\|_{L_t^{q'} L_x^{r'}},\end{aligned}$$

which gives (2.22) in this case. Note that the application of Hardy–Littlewood–Sobolev requires $r<\frac{2(d-1)}{d-3}$. This completes the proof of (2.22) and so the proof of (2.19) and (2.20).

We now turn to (2.21). First we note that by Bernstein's inequality, it suffices to prove the claim for those admissible pairs that are *sharp* admissible in the sense that $\frac{1}{q}+\frac{d-1}{2r}=\frac{d-1}{4}=\frac{1}{\tilde q}+\frac{d-1}{2\tilde r}$. Next, we remark that the proof of (2.22) gives (2.21) for $(\tilde q,\tilde r)=(q,r)$. Finally, to obtain the full range of sharp admissible pairs, one interpolates between this and the following two estimates which are simple consequences of duality and (2.19) and (2.20):

$$\begin{aligned}\Big\| \int_{\mathbb{R}} e^{\pm i(t-s)|\nabla|} P_N F(s)\, ds \Big\|_{L_t^\infty L_x^2} &\lesssim N^{\frac{d}{2}-\frac{1}{\tilde q}-\frac{d}{\tilde r}} \|P_N F\|_{L_t^{\tilde q'} L_x^{\tilde r'}} \\ \Big\| \int_{\mathbb{R}} e^{\pm i(t-s)|\nabla|} P_N F(s)\, ds \Big\|_{L_t^q L_x^r} &\lesssim N^{\frac{d}{2}-\frac{1}{q}-\frac{d}{r}} \|P_N F\|_{L_t^1 L_x^2}.\end{aligned}$$

This completes the proof of the lemma. □

Corollary 2.5 (Strichartz estimates for the half-wave propagator)**.** *Let $d\ge 2$ and (q,r) be wave admissible such that $r\neq\infty$ and $\frac{1}{q}+\frac{d}{r}=\frac{d}{2}-\gamma$ for some $\gamma>0$. Then*

$$\begin{aligned}\|e^{\pm it|\nabla|} f\|_{L_t^q L_x^r} &\lesssim \||\nabla|^\gamma f\|_{L_x^2} \\ \Big\| \int_{\mathbb{R}} e^{\mp it|\nabla|} F(t)\, dt \Big\|_{L_x^2} &\lesssim \||\nabla|^\gamma F\|_{L_t^{q'} L_x^{r'}}.\end{aligned}$$

Moreover, if $(\tilde q,\tilde r)$ is also a wave admissible pair with $\tilde r\neq\infty$, then

$$\Big\| \int_{s<t} e^{\pm i(t-s)|\nabla|} F(s)\, ds \Big\|_{L_t^q L_x^r} \lesssim \||\nabla|^{d-\frac{1}{q}-\frac{1}{\tilde q}-\frac{d}{r}-\frac{d}{\tilde r}} F\|_{L_t^{\tilde q'} L_x^{\tilde r'}}.$$

Proof. In view of Proposition 2.4, it suffices to prove

$$\|F\|_{L_t^q L_x^r} \lesssim \Big\{ \sum_{N\in 2^{\mathbb{Z}}} \|P_N F\|_{L_t^q L_x^r}^2 \Big\}^{1/2} \quad \text{for all } 2\le q\le\infty \text{ and } 2\le r<\infty, \tag{2.23}$$

which by duality is equivalent to

$$\Big\{ \sum_{N\in 2^{\mathbb{Z}}} \|P_N F\|_{L_t^{q'} L_x^{r'}}^2 \Big\}^{1/2} \lesssim \|F\|_{L_t^{q'} L_x^{r'}} \quad \text{for all } 2\le q\le\infty \text{ and } 2\le r<\infty. \tag{2.24}$$

To see that (2.23) and (2.24) are equivalent, consider the operator $T : L_t^{q'} L_x^{r'} \to l^2(L_t^{q'} L_x^{r'})$ given by $T(F) = \{P_N F\}_{N\in 2^{\mathbb{Z}}}$. The operator T being bounded is equivalent to (2.24). It is easy to check that the adjoint of T is $T^* : l^2(L_t^q L_x^r) \to L_t^q L_x^r$ given by $T^*(\{G_N\}_{N\in 2^{\mathbb{Z}}}) = \sum_{N\in 2^{\mathbb{Z}}} P_N G_N$. Boundedness of T^* implies

$$\Big\| \sum_{N\in 2^{\mathbb{Z}}} P_N G_N \Big\|_{L_t^q L_x^r} \lesssim \Big\{ \sum_{N\in 2^{\mathbb{Z}}} \|G_N\|_{L_t^q L_x^r}^2 \Big\}^{1/2}. \tag{2.25}$$

Writing $F = \sum_{N\in 2^{\mathbb{Z}}} P_N F = \sum_{N\in 2^{\mathbb{Z}}} P_N \tilde{P}_N F$ and applying (2.25) with $G_N = \tilde{P}_N F$, we obtain (2.23). Thus (2.24) implies (2.23). To see that (2.23) implies (2.25) and so (2.24), we estimate

$$\begin{aligned}\Big\| \sum_{N\in 2^{\mathbb{Z}}} P_N G_N \Big\|_{L_t^q L_x^r} &\lesssim \Big\{ \sum_{N\in 2^{\mathbb{Z}}} \Big\| P_N \sum_{M\in 2^{\mathbb{Z}}} P_M G_M \Big\|_{L_t^q L_x^r}^2 \Big\}^{1/2} \\ &\lesssim \Big\{ \sum_{N\in 2^{\mathbb{Z}}} \Big\| \sum_{M\sim N} G_M \Big\|_{L_t^q L_x^r}^2 \Big\}^{1/2} \\ &\lesssim \Big\{ \sum_{N\in 2^{\mathbb{Z}}} \|G_N\|_{L_t^q L_x^r}^2 \Big\}^{1/2}.\end{aligned}$$

It thus remains to prove (2.23); for this it suffices to show that

$$\|f\|_{L_x^r} \lesssim \Big\{ \sum_{N\in 2^{\mathbb{Z}}} \|P_N f\|_{L_x^r}^2 \Big\}^{1/2} \quad \text{for all} \quad 2 \le r < \infty, \tag{2.26}$$

since then, for $q \ge 2$ we obtain

$$\begin{aligned}\|F\|_{L_t^q L_x^r} &\lesssim \Big\| \Big\{ \sum_{N\in 2^{\mathbb{Z}}} \|P_N F(t)\|_{L_x^r}^2 \Big\}^{1/2} \Big\|_{L_t^q} = \Big\| \sum_{N\in 2^{\mathbb{Z}}} \|P_N F(t)\|_{L_x^r}^2 \Big\|_{L_t^{\frac{q}{2}}}^{1/2} \\ &\lesssim \Big\{ \sum_{N\in 2^{\mathbb{Z}}} \Big\| \|P_N F(t)\|_{L_x^r}^2 \Big\|_{L_t^{\frac{q}{2}}} \Big\}^{1/2} = \Big\{ \sum_{N\in 2^{\mathbb{Z}}} \|P_N F\|_{L_t^q L_x^r}^2 \Big\}^{1/2}.\end{aligned}$$

Finally, to prove (2.26) we use the square function estimate and the same argument as above:

$$\|f\|_{L_x^r} \sim \Big\| \Big\{ \sum_{N\in 2^{\mathbb{Z}}} |P_N f|^2 \Big\}^{1/2} \Big\|_{L_x^r} \lesssim \Big\{ \sum_{N\in 2^{\mathbb{Z}}} \|P_N f\|_{L_x^r}^2 \Big\}^{1/2} \quad \text{for all} \quad 2 \le r < \infty.$$

This completes the proof of the corollary. □

Corollary 2.6 (Strichartz estimates for the wave equation)**.** *Let $d \ge 2$ and let (q, r) and $(\tilde{q}, \tilde{r})$ be wave admissible pairs such that $r, \tilde{r} < \infty$ and $\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \gamma = \frac{1}{\tilde{q}'} + \frac{d}{\tilde{r}'} - 2$ for some $\gamma > 0$. If u solves*

$$\partial_t^2 u = \Delta u + F \quad \textit{with} \quad u(0) = u_0 \quad \textit{and} \quad \partial_t u(0) = u_1$$

on $I \times \mathbb{R}^d$ for some time interval $I \ni 0$, then

$$\|u\|_{L_t^\infty \dot{H}_x^\gamma} + \|\partial_t u\|_{L_t^\infty \dot{H}_x^{\gamma-1}} + \|u\|_{L_t^q L_x^r} \lesssim \|u_0\|_{\dot{H}_x^\gamma} + \|u_1\|_{\dot{H}_x^{\gamma-1}} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

where all spacetime norms are over $I \times \mathbb{R}^d$.

Proof. Exercise! □

For the Schrödinger equation we have the following Strichartz estimates:

Lemma 2.7 (Strichartz estimates for the Schrödinger equation)**.** *Let $d \geq 1$ and let (q,r) and $(\tilde q, \tilde r)$ be such that $2 \leq q, r, \tilde q, \tilde r \leq \infty$, $\frac{2}{q} + \frac{d}{r} = \frac{d}{2} = \frac{2}{\tilde q} + \frac{d}{\tilde r}$, and $(q,r,d) \neq (2,\infty,2)$ and $(\tilde q, \tilde r, d) \neq (2,\infty,2)$. If u solves*

$$i\partial_t u = -\Delta u + F \quad \textit{with} \quad u(0) = u_0$$

on $I \times \mathbb{R}^d$ for some time interval $I \ni 0$, then

$$\|u\|_{L_t^\infty L_x^2(I\times\mathbb{R}^d)} + \|u\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} \lesssim \|u_0\|_{L_x^2} + \|F\|_{L_t^{\tilde q'} L_x^{\tilde r'}(I\times\mathbb{R}^d)}.$$

Proof. Using as a model the proof of Proposition 2.4, prove the lemma for all pairs of exponents except the endpoints, that is, whenever $r \neq \frac{2d}{d-2}$ and $\tilde r \neq \frac{2d}{d-2}$ for $d \geq 3$. For a proof in the endpoint case, see [17]. $\square$

Finally, we record the Strichartz estimates for the Airy equation:

Lemma 2.8 (Strichartz estimates for the Airy equation)**.** *Let (q,r) and $(\tilde q, \tilde r)$ be such that $2 \leq q, r, \tilde q, \tilde r \leq \infty$, $\frac{1}{q} + \frac{1}{3r} = \frac{1}{6} = \frac{1}{\tilde q} + \frac{1}{3\tilde r}$. If u solves*

$$\partial_t u = -\partial_x^3 u + F \quad \textit{with} \quad u(0) = u_0$$

on $I \times \mathbb{R}$ for some time interval $I \ni 0$, then

$$\|u\|_{L_t^\infty L_x^2(I\times\mathbb{R})} + \|u\|_{L_t^q L_x^r(I\times\mathbb{R})} + \big\||\nabla|^{1/6} u\big\|_{L_{t,x}^6(I\times\mathbb{R})} \lesssim \|u_0\|_{L_x^2} + \|F\|_{L_t^{\tilde q'} L_x^{\tilde r'}(I\times\mathbb{R})}.$$

Proof. Exercise! $\square$

2.5. **Bilinear Strichartz and local smoothing estimates.** In this subsection, we restrict attention to the Schrödinger propagator.

Theorem 2.9 (Bilinear Strichartz I, [3, 13, 28])**.** *Fix $d \geq 1$ and $M \leq N$. Then*

$$\big\|[e^{it\Delta} P_M f][e^{it\Delta} P_N g]\big\|_{L_{t,x}^2(\mathbb{R}\times\mathbb{R}^d)} \lesssim M^{\frac{d-1}{2}} N^{-\frac12} \|f\|_{L_x^2(\mathbb{R}^d)} \|g\|_{L_x^2(\mathbb{R}^d)}. \tag{2.27}$$

When $d = 1$ we require $M \leq \frac14 N$, so that $P_N P_M = 0$.

Proof. For $M \sim N$ and $d \neq 1$, the result follows from the $L_x^2 \to L_t^4 L_x^{\frac{2d}{d-1}}$ Strichartz inequality and Bernstein.

Turning to the case $M \leq \frac14 N$, we note that by duality and the Parseval identity, it suffices to show

$$\begin{aligned} &\Big|\iint_{\mathbb{R}^d\times\mathbb{R}^d} F(|\xi|^2 + |\eta|^2, \xi + \eta) \widehat{f_M}(\xi) \widehat{g_N}(\eta)\, d\xi\, d\eta\Big| \\ &\qquad\qquad \lesssim M^{\frac{d-1}{2}} N^{-\frac12} \|F\|_{L_{\omega,\xi}^2(\mathbb{R}^{1+d})} \|\hat f\|_{L_\xi^2(\mathbb{R}^d)} \|\hat g\|_{L_\xi^2(\mathbb{R}^d)}. \end{aligned} \tag{2.28}$$

By breaking the region of integration into several pieces (and rotating the coordinate system appropriately), we may restrict the region of integration to a set where $\eta_1 - \xi_1 \gtrsim N$. Next, we make the change of variables $\zeta = \xi + \eta$, $\omega = |\xi|^2 + |\eta|^2$, and $\beta = (\xi_2, \ldots, \xi_d)$. Note that $|\beta| \lesssim M$ while the Jacobian is $J \sim N^{-1}$. Using this information together with Cauchy–Schwarz:

$$\begin{aligned} \text{LHS(2.28)} &= \Big|\iiint F(\omega,\zeta) \widehat{f_M}(\xi) \widehat{g_N}(\eta) J\, d\omega\, d\zeta\, d\beta\Big| \\ &\leq \|F\|_{L_{\omega,\xi}^2(\mathbb{R}^{1+d})} \int \Big[\iint |\widehat{f_M}(\xi)|^2 |\widehat{g_N}(\eta)|^2 J^2\, d\omega\, d\zeta\Big]^{\frac12} d\beta \\ &\lesssim \|F\|_{L_{\omega,\xi}^2(\mathbb{R}^{1+d})} M^{\frac{d-1}{2}} \Big(\iiint |\widehat{f_M}(\xi)|^2 |\widehat{g_N}(\eta)|^2 J^2\, d\omega\, d\zeta\, d\beta\Big)^{\frac12} \end{aligned}$$

$$\lesssim \|F\|_{L^2_{\omega,\xi}(\mathbb{R}^{1+d})} M^{\frac{d-1}{2}} \Big(\iint |\widehat{f_M}(\xi)|^2 |\widehat{g_N}(\eta)|^2 N^{-1}\, d\xi\, d\eta \Big)^{\frac12},$$

which implies (2.27). □

Corollary 2.10 (Bilinear Strichartz II). *Let M, N, and d be as above. Given any spacetime slab $I \times \mathbb{R}^d$ and any functions u, v defined on $I \times \mathbb{R}^d$,*

$$\big\| u_{\le M} v_{\ge N} \big\|_{L^2_{t,x}(I\times\mathbb{R}^d)} \lesssim M^{\frac{d-3}{2}} N^{-\frac12} \|\nabla u_{\le M}\|_{S_0^*(I)} \|v_{\ge N}\|_{S_0^*(I)},$$

where we use the notation

$$\|u\|_{S_0^*(I)} := \|u\|_{L^\infty_t L^2_x(I\times\mathbb{R}^d)} + \|(i\partial_t + \Delta)u\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I\times\mathbb{R}^d)}.$$

Proof. See [39, Lemma 2.5], which builds on earlier versions in [4, 13]. □

Lemma 2.11 (Local smoothing, [14, 32, 38]). *For all $f \in L^2_x$ we have*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \big| \big[|\nabla|^{1/2} e^{it\Delta} f \big](x) \big|^2 e^{-|x|^2}\, dx\, dt \lesssim \|f\|_{L^2_x}^2.$$

In particular, by scaling, for all $R > 0$ we have

$$\big\| |\nabla|^{1/2} e^{it\Delta} f \big\|_{L^2_{t,x}(\mathbb{R}\times B(0,R))} \lesssim R^{1/2} \|f\|_{L^2_x}.$$

Proof. Given $a : \mathbb{R}^d \to [0,\infty)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}^d} \big| \big[|\nabla|^{1/2} e^{it\Delta} f \big](x) \big|^2 a(x)\, dx\, dt \\
&\quad = (2\pi)^{-d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{ix\xi - it|\xi|^2} |\xi|^{1/2} \hat f(\xi) e^{-ix\eta + it|\eta|^2} |\eta|^{1/2} \overline{\hat f(\eta)} a(x)\, d\xi\, d\eta\, dx\, dt \\
&\quad = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat a(\eta - \xi) \delta(|\eta|^2 - |\xi|^2) |\xi|^{1/2} |\eta|^{1/2} \hat f(\xi) \overline{\hat f(\eta)}\, d\xi\, d\eta \\
&\quad = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat a(\eta - \xi) \delta(|\eta| - |\xi|) \frac{|\xi|^{1/2}|\eta|^{1/2}}{|\xi| + |\eta|} \hat f(\xi) \overline{\hat f(\eta)}\, d\xi\, d\eta.
\end{aligned}$$

By Schur's test it thus suffices to show

$$\int_{\mathbb{R}^d} \hat a(\eta - \xi) \delta(|\eta| - |\xi|) \frac{|\xi|^{1/2}|\eta|^{1/2}}{|\xi| + |\eta|}\, d\xi \lesssim 1 \quad \text{uniformly in} \quad \eta \in \mathbb{R}^d. \tag{2.29}$$

Recalling that in our case $a(x) = e^{-|x|^2}$ and passing to polar coordinates, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \hat a(\eta - \xi) \delta(|\eta| - |\xi|) \frac{|\xi|^{1/2}|\eta|^{1/2}}{|\xi| + |\eta|}\, d\xi \\
&\quad \lesssim \int_{S^{d-1}} \int_0^\infty e^{-|r\omega - \eta|^2} \delta(|\eta| - r) \frac{r^{1/2}|\eta|^{1/2}}{r + |\eta|} r^{d-1}\, dr\, d\sigma(\omega) \\
&\quad \lesssim \int_{S^{d-1}} \int_0^\infty e^{-|\eta|^2 \left|\omega - \frac{\eta}{|\eta|}\right|^2} |\eta|^{d-1}\, d\sigma(\omega) \\
&\quad \lesssim \int_0^\pi e^{-2|\eta|^2(1-\cos\theta)} |\eta|^{d-1} (\sin\theta)^{d-2}\, d\theta \\
&\quad \lesssim \int_0^{\frac{\pi}{2}} e^{-\frac{|\eta|^2\theta^2}{100}} |\eta|^{d-1} \theta^{d-2}\, d\theta \lesssim \int_0^\infty e^{-\frac{\tau^2}{100}} \tau^{d-2}\, d\tau \lesssim 1.
\end{aligned}$$

In the computation above, θ denotes the angle ω makes with $\frac{\eta}{|\eta|}$. This proves (2.29) and so completes the proof of the lemma. $\square$

The next result is a consequence of local smoothing; see Lemma 3.7 in [18]. The proof we present here is the one from [23]; see also [22].

Lemma 2.12. *Given $\phi \in \dot{H}^1(\mathbb{R}^d)$,*

$$\|\nabla e^{it\Delta}\phi\|^3_{L^2_{t,x}([-T,T]\times\{|x|\le R\})} \lesssim T^{\frac{2}{d+2}} R^{\frac{3d+2}{d+2}} \|e^{it\Delta}\phi\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \|\nabla\phi\|^2_{L^2_x}.$$

Proof. Given $N > 0$, Hölder's and Bernstein's inequalities imply

$$\begin{aligned}\|\nabla e^{it\Delta}\phi_{<N}\|_{L^2_{t,x}([-T,T]\times\{|x|\le R\})} &\lesssim T^{\frac{2}{d+2}} R^{\frac{2d}{d+2}} \|e^{it\Delta}\nabla\phi_{<N}\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \\ &\lesssim T^{\frac{2}{d+2}} R^{\frac{2d}{d+2}}\, N\, \|e^{it\Delta}\phi\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}.\end{aligned}$$

On the other hand, the high frequencies can be estimated using local smoothing:

$$\begin{aligned}\|\nabla e^{it\Delta}\phi_{\ge N}\|_{L^2_{t,x}([-T,T]\times\{|x|\le R\})} &\lesssim R^{1/2}\||\nabla|^{1/2}\phi_{\ge N}\|_{L^2_x} \\ &\lesssim N^{-1/2}R^{1/2}\|\nabla\phi\|_{L^2_x}.\end{aligned}$$

The lemma now follows by optimizing the choice of N. $\square$

3. An inverse Strichartz inequality

In this section, we develop tools that we will employ to prove a linear profile decomposition for the Schrödinger propagator for bounded sequences in $\dot{H}^1(\mathbb{R}^d)$ with $d \ge 3$. Such a linear profile decomposition was first obtained by Keraani [18], relying on an improved Sobolev inequality proved by Gerard, Meyer, and Oru [16]. We should also note the influential precursor [1] which treated the wave equation. In these notes we present a different proof of the result in [18], which relies instead on an inverse Strichartz inequality.

A linear profile decomposition for the Schrödinger propagator for bounded sequences in $L^2(\mathbb{R}^d)$ was proved by Merle–Vega [26] for $d = 2$, Carles–Keraani [7] for $d = 1$, and Bégout–Vargas [2] for $d \ge 3$. For a different approach to these results, which is similar in spirit to what we present in these notes, see [22].

We start by noting that combining the Strichartz inequality for the Schrödinger propagator from Lemma 2.7 and Sobolev embedding, we obtain

$$\|e^{it\Delta}f\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|e^{it\Delta}\nabla f\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+8}}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|f\|_{\dot{H}^1_x} \tag{3.1}$$

for all $d \ge 3$.

Our next result is a refinement of (3.1), which says that if the linear evolution of f is large in $L_{t,x}^{\frac{2(d+2)}{d-2}}$, then the linear evolution of a single Littlewood–Paley piece of f is, at least partially, responsible.

Lemma 3.1 (Refined Strichartz estimate)**.** *Let $d \ge 3$ and $f \in \dot{H}^1(\mathbb{R}^d)$. Then*

$$\|e^{it\Delta}f\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|f\|^{\frac{d-2}{d+2}}_{\dot{H}^1_x} \sup_{N\in 2^{\mathbb{Z}}} \|e^{it\Delta}f_N\|^{\frac{4}{d+2}}_{L_{t,x}^{\frac{2(d+2)}{d-2}}(\mathbb{R}\times\mathbb{R}^d)}.$$

Proof. We will present the proof in dimensions $d \geq 6$. The proof in dimensions $d = 3, 4$ is easier as $\frac{2(d+2)}{d-2}$ is an even integer in those cases. The proof in dimension $d = 5$ is a small modification of the argument below. We leave the cases $d = 3, 4, 5$ as an exercise for the conscientious reader.

Fix $d \geq 6$. From the square function estimate, the subaditivity of fractional powers (using the fact that $\frac{d+2}{2(d-2)} \leq 1$ in dimensions $d \geq 6$), and the Bernstein and Strichartz inequalities,

$$
\begin{aligned}
&\|e^{it\Delta} f\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{2(d+2)}{d-2}} \\
&\quad \lesssim \iint_{\mathbb{R}\times\mathbb{R}^d} \Big(\sum_{N \in 2^{\mathbb{Z}}} |e^{it\Delta} f_N|^2 \Big)^{\frac{d+2}{d-2}}\, dx\, dt \\
&\quad \lesssim \sum_{M \leq N} \iint_{\mathbb{R}\times\mathbb{R}^d} |e^{it\Delta} f_M|^{\frac{d+2}{d-2}} |e^{it\Delta} f_M|^{\frac{d+2}{d-2}}\, dx\, dt \\
&\quad \lesssim \sum_{M \leq N} \|e^{it\Delta} f_M\|_{L_{t,x}^{\frac{2(d+2)}{d-4}}} \|e^{it\Delta} f_M\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}} \|e^{it\Delta} f_N\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}} \|e^{it\Delta} f_N\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \\
&\quad \lesssim \sup_{N \in 2^{\mathbb{Z}}} \|e^{it\Delta} f_N\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{8}{d-2}} \sum_{M \leq N} M^2 \|e^{it\Delta} f_M\|_{L_t^{\frac{2(d+2)}{d-4}} L_x^{\frac{2d(d+2)}{d^2+8}}} \|f_N\|_{L_x^2} \\
&\quad \lesssim \sup_{N \in 2^{\mathbb{Z}}} \|e^{it\Delta} f_N\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{8}{d-2}} \sum_{M \leq N} M^2 \|f_M\|_{L_x^2} \|f_N\|_{L_x^2} \\
&\quad \lesssim \sup_{N \in 2^{\mathbb{Z}}} \|e^{it\Delta} f_N\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{8}{d-2}} \sum_{M \leq N} \tfrac{M}{N} \|\nabla f_M\|_{L_x^2} \|\nabla f_N\|_{L_x^2} \\
&\quad \lesssim \sup_{N \in 2^{\mathbb{Z}}} \|e^{it\Delta} f_N\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{8}{d-2}} \|f\|_{\dot H_x^1}^2 .
\end{aligned}
$$

This completes the proof of the lemma in dimensions $d \geq 6$. □

The refined Strichartz inequality shows that linear solutions with non-trivial spacetime norm must concentrate on at least one frequency annulus. The next proposition goes one step further and shows that they contain a bubble of concentration around some point in spacetime.

Proposition 3.2 (Inverse Strichartz inequality)**.** *Let $d \geq 3$ and let $\{f_n\} \subset \dot H^1(\mathbb{R}^d)$. Suppose that*

$$\lim_{n\to\infty} \|f_n\|_{\dot H_x^1} = A < \infty \quad \text{and} \quad \lim_{n\to\infty} \|e^{it\Delta} f_n\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(\mathbb{R}\times\mathbb{R}^d)} = \varepsilon > 0.$$

Then there exist a subsequence in n, $\phi \in \dot H_x^1$, $\{\lambda_n\} \subset (0,\infty)$, and $\{(t_n, x_n)\} \subset \mathbb{R}\times\mathbb{R}^d$ such that

$$\lambda_n^{\frac{d-2}{2}} [e^{it_n\Delta} f_n](\lambda_n x + x_n) \rightharpoonup \phi(x) \quad \text{weakly in} \quad \dot H_x^1, \tag{3.2}$$

$$\liminf_{n\to\infty} \Big\{ \|f_n\|_{\dot H_x^1}^2 - \|f_n - \phi_n\|_{\dot H_x^1}^2 \Big\} = \|\phi\|_{\dot H_x^1}^2 \gtrsim A^2 (\tfrac{\varepsilon}{A})^{\frac{d(d+2)}{4}}, \tag{3.3}$$

$$\liminf_{n\to\infty} \Big\{ \|e^{it\Delta} f_n\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{2(d+2)}{d-2}} - \|e^{it\Delta}(f_n - \phi_n)\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{2(d+2)}{d-2}} \Big\} \gtrsim \varepsilon^{\frac{2(d+2)}{d-2}} (\tfrac{\varepsilon}{A})^{\frac{(d+2)(d+4)}{4}}, \tag{3.4}$$

where

$$\phi_n(x) := \lambda_n^{-\frac{d-2}{2}} [e^{-i\lambda_n^{-2} t_n \Delta}\phi]\big(\tfrac{x-x_n}{\lambda_n}\big). \tag{3.5}$$

Proof. Passing to a subsequence, we may assume

$$\lim_{n\to\infty} \|f_n\|_{\dot H^1_x} \leq 2A \quad \text{and} \quad \lim_{n\to\infty} \|e^{it\Delta} f_n\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(\mathbb{R}\times\mathbb{R}^d)} \geq \tfrac{\varepsilon}{2}.$$

Thus, using Lemma 3.1 we see that for each n there exists $N_n \in 2^{\mathbb{Z}}$ such that

$$\|e^{it\Delta} P_{N_n} f_n\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \gtrsim \varepsilon^{\frac{d+2}{4}} A^{-\frac{d-2}{4}}.$$

On the other hand, from the Strichartz and Bernstein inequalities we get

$$\|e^{it\Delta} P_{N_n} f_n\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \lesssim \|P_{N_n} f_n\|_{L^2_x} \lesssim N_n^{-1} A.$$

By Hölder's inequality, these imply

$$\begin{aligned}
\varepsilon^{\frac{d+2}{4}} A^{-\frac{d-2}{4}} &\lesssim \|e^{it\Delta} P_{N_n} f_n\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \\
&\lesssim \|e^{it\Delta} P_{N_n} f_n\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{d-2}{d}} \|e^{it\Delta} P_{N_n} f_n\|_{L^\infty_{t,x}}^{\frac{2}{d}} \\
&\lesssim N_n^{-\frac{d-2}{d}} A^{\frac{d-2}{d}} \|e^{it\Delta} P_{N_n} f_n\|_{L^\infty_{t,x}}^{\frac{2}{d}},
\end{aligned}$$

and so

$$N_n^{-\frac{d-2}{2}} \|e^{it\Delta} P_{N_n} f_n\|_{L^\infty_{t,x}} \gtrsim A\big(\tfrac{\varepsilon}{A}\big)^{\frac{d(d+2)}{8}}.$$

Thus there exist $(t_n, x_n) \in \mathbb{R}\times\mathbb{R}^d$ such that

$$N_n^{-\frac{d-2}{2}} \big|[e^{it_n\Delta} P_{N_n} f_n](x_n)\big| \gtrsim A\big(\tfrac{\varepsilon}{A}\big)^{\frac{d(d+2)}{8}}. \tag{3.6}$$

We define the spatial scales $\lambda_n := N_n^{-1}$.

It remains to find the profile ϕ and to prove it satisfies (3.2) through (3.4). To this end, we set

$$g_n(x) := \lambda_n^{\frac{d-2}{2}} [e^{it_n\Delta} f_n](\lambda_n x + x_n).$$

A simple change of variables gives

$$\|g_n\|_{\dot H^1_x} = \|f_n\|_{\dot H^1_x} \lesssim A$$

and so, passing to a subsequence, we can choose ϕ so that $g_n \rightharpoonup \phi$ weakly in $\dot H^1_x$. This proves (3.2).

We now turn to (3.3). The asymptotic decoupling statement is immediate since $\dot H^1_x$ is a Hilbert space. We are left to prove the lower bound in (3.3). Toward this end, let $\check\psi := P_1\delta_0$ denote the convolution kernel associated with P_1. Using a change of variables and (3.6), we get

$$\begin{aligned}
\big|\langle \phi, \check\psi\rangle_{L^2_x}\big| = \big|\lim_{n\to\infty} \langle g_n, \check\psi\rangle_{L^2_x}\big| &= \big|\lim_{n\to\infty} \big\langle e^{it_n\Delta} f_n, \lambda_n^{-\frac{d+2}{2}} \check\psi\big(\tfrac{x-x_n}{\lambda_n}\big)\big\rangle_{L^2_x}\big| \\
&= N_n^{-\frac{d-2}{2}} \big|[e^{it_n\Delta} P_{N_n} f_n](x_n)\big| \gtrsim A\big(\tfrac{\varepsilon}{A}\big)^{\frac{d(d+2)}{8}}.
\end{aligned} \tag{3.7}$$

On the other hand, by Hölder's inequality and Sobolev embedding,

$$\big|\langle \phi, \check\psi\rangle_{L^2_x}\big| \lesssim \|\phi\|_{L^6_x} \|\check\psi\|_{L_x^{6/5}} \lesssim \|\phi\|_{\dot H^1_x}.$$

Putting the two inequalities together, we derive the lower bound in (3.3).

It remains to prove (3.4). We start by proving decoupling for the $L_{t,x}^{\frac{2(d+2)}{d-2}}$ norm. Note that

$$(i\partial_t)^{\frac{1}{2}} e^{it\Delta} = (-\Delta)^{\frac{1}{2}} e^{it\Delta},$$

as can be checked by testing against Schwartz functions in $\mathbb{R}\times\mathbb{R}^d$. Thus, by Hölder's inequality, on any compact set K in $\mathbb{R}\times\mathbb{R}^d$ we have

$$\|e^{it\Delta} g_n\|_{H^{\frac{1}{2}}_{t,x}(K)} \lesssim \|\langle -\Delta\rangle^{\frac{1}{2}} e^{it\Delta} g_n\|_{L^2_{t,x}(K)} \lesssim_K A.$$

Using this together with Rellich–Kondrashov and passing to a subsequence, we get

$$e^{it\Delta} g_n \to e^{it\Delta}\phi \quad \text{strongly in} \quad L^2_{t,x}(K).$$

(In order to identify the limit in the display above, we note that $g_n \rightharpoonup \phi$ weakly in $\dot{H}^1_x$ implies that $e^{it\Delta} g_n$ converges to $e^{it\Delta}\phi$ as distributions on $\mathbb{R}\times\mathbb{R}^d$.) Passing to a further subsequence, we deduce that $e^{it\Delta} g_n \to e^{it\Delta}\phi$ a.e. on K. Finally, using a diagonal argument and passing again to a subsequence if necessary, we obtain

$$e^{it\Delta} g_n \to e^{it\Delta}\phi \quad \text{a.e. in } \mathbb{R}\times\mathbb{R}^d.$$

To continue, we use this convergence together with the refined Fatou lemma (see Lemma A.3) due to Brézis and Lieb and a change of variables; we obtain

$$\lim_{n\to\infty}\Big\{\|e^{it\Delta} f_n\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{2(d+2)}{d-2}} - \|e^{it\Delta}(f_n-\phi_n)\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{2(d+2)}{d-2}}\Big\} = \|e^{it\Delta}\phi\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{2(d+2)}{d-2}},$$

from which (3.4) will follow once we prove

$$\|e^{it\Delta}\phi\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \gtrsim \varepsilon\big(\tfrac{\varepsilon}{A}\big)^{\frac{d^2+2d-8}{8}}. \tag{3.8}$$

To see this, we use (3.7), the Mikhlin multiplier theorem, and Bernstein to estimate

$$\begin{aligned} A\big(\tfrac{\varepsilon}{A}\big)^{\frac{d(d+2)}{8}} \lesssim \big|\langle \phi, \check{\psi}\rangle_{L^2_x}\big| = \big|\langle e^{it\Delta}\phi, e^{it\Delta}\check{\psi}\rangle_{L^2_x}\big| &\lesssim \|e^{it\Delta}\phi\|_{L_x^{\frac{2(d+2)}{d-2}}} \|e^{it\Delta}\check{\psi}\|_{L_x^{\frac{2(d+2)}{d+6}}} \\ &\lesssim \|e^{it\Delta}\phi\|_{L_x^{\frac{2(d+2)}{d-2}}}, \end{aligned}$$

uniformly in $|t|\le 1$. Integrating in t leads to (3.8). □

Exercise 3.1. Under the hypotheses of Proposition 3.2 and passing to a further subsequence if necessary, prove decoupling of the potential energy, namely,

$$\liminf_{n\to\infty}\Big\{\|f_n\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}} - \|f_n-\phi_n\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}} - \|e^{-i\lambda_n^{-2}t_n\Delta}\phi\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}}\Big\} = 0.$$

Hint: Passing to a subsequence, we may assume that $\lambda_n^{-2}t_n \to t_0 \in [-\infty,\infty]$. If $t_0=\pm\infty$, then approximate ϕ in $\dot{H}^1_x$ by Schwartz functions and use the fact that by the dispersive estimate for the Schrödinger propagator,

$$\|e^{-i\lambda_n^{-2}t_n\Delta}\psi\|_{\frac{2d}{d-2}} \to 0 \quad \text{as} \quad n\to\infty$$

for any $\psi\in\mathcal{S}(\mathbb{R}^d)$. If instead $t_0\in(-\infty,\infty)$, then (3.2) can be upgraded to $\lambda_n^{\frac{d-2}{2}} f_n(\lambda_n x + x_n) \rightharpoonup e^{-it_0\Delta}\phi(x)$ weakly in $\dot{H}^1_x$. Now use Rellich–Kondrashov and refined Fatou as in the proof of (3.4).

4. A linear profile decomposition

In this section, we use the inverse Strichartz inequality Proposition 3.2 to derive a linear profile decomposition for the Schrödinger propagator.

One can view the linear profile decomposition as a tool for measuring the defects of compactness in the Strichartz inequality (3.1). More precisely, given a bounded sequence of functions $\{f_n\}_{n\geq 1} \subset \dot H^1(\mathbb{R}^d)$ we would like to be able to say that, after possibly passing to a subsequence, $\{e^{it\Delta} f_n\}_{n\geq1}$ converges in $L_{t,x}^{\frac{2(d+2)}{d-2}}$. Unfortunately, every non-compact symmetry of the inequality (3.1) is a reason why we would fail to extract a convergent subsequence.

The non-compact symmetries of (3.1) are space- and time-translations and $\dot H^1_x$-preserving scaling. To see how these work against us, consider the simple scenario where $f_n(x) = f(x+x_n)$ with $f\in \dot H^1_x$ and $\{x_n\}_{n\geq1}\subset\mathbb{R}^d$ is a sequence that diverges to infinity; in this case, $\{e^{it\Delta} f_n\}_{n\geq1}$ converges weakly to zero. We leave it to the reader to use time-translations and $\dot H^1_x$-preserving scaling to construct bounded sequences of functions $\{f_n\}_{n\geq1}\subset \dot H^1(\mathbb{R}^d)$ for which $\{e^{it\Delta} f_n\}_{n\geq1}$ converges weakly to zero.

At this point we might imagine that if suitably translate and rescale our sequence, then we might be able to extract a convergent subsequence. Proposition 3.2 gives us hope, since it exhibits a bubble of concentration living inside each $e^{it\Delta} f_n$, which captures a nontrivial portion of the $L_{t,x}^{\frac{2(d+2)}{d-2}}$ norm of $e^{it\Delta} f_n$. However, even this modified goal is naive and doomed to fail, as one can see by considering the following scenario: $f_n(x) = f(x) + f(x+x_n)$ with $f\in\dot H^1_x$ and $\{x_n\}_{n\geq1}\subset\mathbb{R}^d$ is a sequence that diverges to infinity; in this case, the evolutions $e^{it\Delta} f_n$ contain two diverging bubbles of concentration and translating our sequence would still fail to exhibit a convergent subsequence.

Nevertheless, this suggests that if we take out enough bubbles of concentration living inside $e^{it\Delta} f_n$, then we might be able to say that the remainders do indeed converge to zero in $L_{t,x}^{\frac{2(d+2)}{d-2}}$. This is precisely the content of the following theorem.

Theorem 4.1 ($\dot H^1_x$ linear profile decomposition for the Schrödinger propagator)**.** *Fix $d\geq 3$ and let $\{f_n\}_{n\geq1}$ be a sequence of functions bounded in $\dot H^1(\mathbb{R}^d)$. Passing to a subsequence if necessary, there exist $J^*\in\{0,1,\ldots\}\cup\{\infty\}$, functions $\{\phi^j\}_{j=1}^{J^*}\subset \dot H^1(\mathbb{R}^d)$, $\{\lambda_n^j\}\subset(0,\infty)$, and $\{t_n^j, x_n^j\}\subset\mathbb{R}\times\mathbb{R}^d$ such that for each finite $0\leq J\leq J^*$, we have the decomposition*

$$f_n = \sum_{j=1}^J (\lambda_n^j)^{-\frac{d-2}{2}} [e^{it_n^j\Delta}\phi^j]\big(\tfrac{x-x_n^j}{\lambda_n^j}\big) + w_n^J \tag{4.1}$$

with the following properties:

$$\lim_{J\to J^*}\limsup_{n\to\infty} \big\| e^{it\Delta} w_n^J\big\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(\mathbb{R}\times\mathbb{R}^d)} = 0 \tag{4.2}$$

$$\lim_{n\to\infty}\Big[\|\nabla f_n\|_2^2 - \sum_{j=1}^J \|\nabla\phi^j\|_2^2 - \|\nabla w_n^J\|_2^2\Big] = 0 \tag{4.3}$$

$$\lim_{n\to\infty}\Big[\|f_n\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}} - \sum_{j=1}^J \|e^{it_n^j\Delta}\phi^j\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}} - \|w_n^J\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}}\Big] = 0 \tag{4.4}$$

$$e^{-it_n^J\Delta}\big[(\lambda_n^J)^{\frac{d-2}{2}}w_n^J(\lambda_n^J x+x_n^J)\big]\rightharpoonup 0 \quad \textit{weakly in } \dot H^1(\mathbb{R}^d). \tag{4.5}$$

Moreover, for each $j\neq k$ we have the following asymptotic decoupling of parameters:

$$\frac{\lambda_n^j}{\lambda_n^k}+\frac{\lambda_n^k}{\lambda_n^j}+\frac{|x_n^j-x_n^k|^2}{\lambda_n^j\lambda_n^k}+\frac{\big|t_n^j(\lambda_n^j)^2-t_n^k(\lambda_n^k)^2\big|}{\lambda_n^j\lambda_n^k}\to\infty \quad as \quad n\to\infty. \tag{4.6}$$

Lastly, we may additionally assume that for each j either $t_n^j\equiv 0$ or $t_n^j\to\pm\infty$.

Proof. To keep formulas within margins, we will use the notation

$$(g_n^j f)(x):=(\lambda_n^j)^{-\frac{d-2}{2}}f\big(\tfrac{x-x_n^j}{\lambda_n^j}\big) \quad \text{with} \quad [(g_n^j)^{-1}f](x):=(\lambda_n^j)^{\frac{d-2}{2}}f\big(\lambda_n^j x+x_n^j\big).$$

Note that $\|g_n^j f\|_{\dot H_x^1}=\|f\|_{\dot H_x^1}=\|(g_n^j)^{-1}f\|_{\dot H_x^1}$ and

$$\langle g_n^j f_1,f_2\rangle_{\dot H_x^1}=\langle f_1,(g_n^j)^{-1}f_2\rangle_{\dot H_x^1} \quad \text{for all} \quad f_1,f_2\in\dot H_x^1.$$

We will also use the notation

$$\phi_n^j(x):=(\lambda_n^j)^{-\frac{d-2}{2}}[e^{it_n^j\Delta}\phi^j]\big(\tfrac{x-x_n^j}{\lambda_n^j}\big)=[g_n^j e^{it_n^j\Delta}\phi^j](x).$$

To prove the theorem we will proceed inductively, extracting one bubble at a time. To start, we set $w_n^0:=f_n$. Now suppose we have a decomposition up to level $J\ge 0$ obeying (4.3) through (4.5). (Conditions (4.2) and (4.6) will be verified at the end.) Passing to a subsequence if necessary, we set

$$A_J:=\lim_{n\to\infty}\|w_n^J\|_{\dot H_x^1} \quad \text{and} \quad \varepsilon_J:=\lim_{n\to\infty}\|e^{it\Delta}w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}.$$

If $\varepsilon_J=0$, we stop and set $J^*=J$. If not, we apply Proposition 3.2 to w_n^J. Thus, passing to a subsequence in n, we find $\phi^{J+1}\in\dot H_x^1$, $\{\lambda_n^{J+1}\}\subset(0,\infty)$, and $\{(t_n^{J+1},x_n^{J+1})\}\subset\mathbb{R}\times\mathbb{R}^d$, where we renamed the time parameters given by Proposition 3.2 as follows: $t_n^{J+1}=-\lambda_n^{-2}t_n$.

According to Proposition 3.2, the profile ϕ^{J+1} is defined as a weak limit, namely,

$$\phi^{J+1}=\operatorname*{w-lim}_{n\to\infty}(g_n^{J+1})^{-1}\big[e^{-it_n^{J+1}(\lambda_n^{J+1})^2\Delta}w_n^J\big]=\operatorname*{w-lim}_{n\to\infty}e^{-it_n^{J+1}\Delta}[(g_n^{J+1})^{-1}w_n^J].$$

We let $\phi_n^{J+1}:=g_n^{J+1}e^{it_n^{J+1}\Delta}\phi^{J+1}$.

Now define $w_n^{J+1}:=w_n^J-\phi_n^{J+1}$. By the definition of ϕ^{J+1},

$$e^{-it_n^{J+1}\Delta}(g_n^{J+1})^{-1}w_n^{J+1}\rightharpoonup 0 \quad \text{weakly in } \dot H_x^1.$$

This proves (4.5) at the level $J+1$. Moreover, from Proposition 3.2 we also have

$$\lim_{n\to\infty}\Big\{\|w_n^J\|_{\dot H_x^1}^2-\|w_n^{J+1}\|_{\dot H_x^1}^2-\|\phi^{J+1}\|_{\dot H_x^1}^2\Big\}=0.$$

Combining this with the inductive hypothesis gives (4.3) at the level $J+1$. A similar argument using Exercise 3.1 establishes (4.4) at the same level.

Passing to a further subsequence and using Proposition 3.2, we obtain

$$\begin{aligned} A_{J+1}^2&=\lim_{n\to\infty}\|w_n^{J+1}\|_{\dot H_x^1}^2\le A_J^2\Big[1-C\big(\tfrac{\varepsilon_J}{A_J}\big)^{\frac{d(d+2)}{4}}\Big]\le A_J^2\\ \varepsilon_{J+1}^{\frac{2(d+2)}{d-2}}&=\lim_{n\to\infty}\|e^{it\Delta}w_n^{J+1}\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{2(d+2)}{d-2}}\le\varepsilon_J^{\frac{2(d+2)}{d-2}}\Big[1-C\big(\tfrac{\varepsilon_J}{A_J}\big)^{\frac{(d+2)(d+4)}{4}}\Big].\end{aligned} \tag{4.7}$$

If $\varepsilon_{J+1}=0$ we stop and set $J^*=J+1$; in this case, (4.2) is automatic. If $\varepsilon_{J+1}>0$ we continue the induction. If the algorithm does not terminate in finitely many steps, we set $J^* = \infty$; in this case, (4.7) implies $\varepsilon_J \to 0$ as $J \to \infty$ and so (4.2) follows.

Next we verify the asymptotic orthogonality condition (4.6). We argue by contradiction. Assume (4.6) fails to be true for some pair (j,k). Without loss of generality, we may assume that this is the first pair for which (4.6) fails, that is, $j < k$ and (4.6) holds for all pairs (j,l) with $j < l < k$. Passing to a subsequence, we may assume

$$\frac{\lambda_n^j}{\lambda_n^k} \to \lambda_0 \in (0,\infty), \quad \frac{x_n^j - x_n^k}{\sqrt{\lambda_n^j \lambda_n^k}} \to x_0, \quad \text{and} \quad \frac{t_n^j(\lambda_n^j)^2 - t_n^k(\lambda_n^k)^2}{\lambda_n^j\lambda_n^k} \to t_0. \tag{4.8}$$

From the inductive relation

$$w_n^{k-1} = w_n^j - \sum_{l=j+1}^{k-1} \phi_n^l$$

and the definition of ϕ^k, we obtain

$$\begin{aligned}\phi^k &= \operatorname*{w-lim}_{n\to\infty} e^{-it_n^k\Delta}[(g_n^k)^{-1}w_n^{k-1}]\\ &= \operatorname*{w-lim}_{n\to\infty} e^{-it_n^k\Delta}[(g_n^k)^{-1}w_n^j] - \sum_{l=j+1}^{k-1} \operatorname*{w-lim}_{n\to\infty} e^{-it_n^k\Delta}[(g_n^k)^{-1}\phi_n^l]. \end{aligned}\tag{4.9}$$

We will prove that these weak limits are all zero and so obtain a contradiction to the nontriviality of ϕ^k.

We write

$$\begin{aligned}e^{-it_n^k\Delta}[(g_n^k)^{-1}w_n^j] &= e^{-it_n^k\Delta}(g_n^k)^{-1}g_n^j e^{it_n^j\Delta}[e^{-it_n^j\Delta}(g_n^j)^{-1}w_n^j]\\ &= (g_n^k)^{-1}g_n^j e^{i\left(t_n^j - t_n^k\frac{(\lambda_n^k)^2}{(\lambda_n^j)^2}\right)\Delta}[e^{-it_n^j\Delta}(g_n^j)^{-1}w_n^j].\end{aligned}$$

Note that by (4.8),

$$t_n^j - t_n^k\frac{(\lambda_n^k)^2}{(\lambda_n^j)^2} = \frac{t_n^j(\lambda_n^j)^2 - t_n^k(\lambda_n^k)^2}{\lambda_n^j\lambda_n^k}\cdot\frac{\lambda_n^k}{\lambda_n^j} \to \frac{t_0}{\lambda_0}.$$

Using this together with (4.5), Exercise 4.2, and the fact that the adjoints of the unitary operators $(g_n^k)^{-1}g_n^j$ converge strongly, we obtain that the first term on RHS(4.9) is zero.

To complete the proof of (4.6), it remains to show that the second term on RHS(4.9) is zero. For all $j < l < k$ we write

$$e^{-it_n^k\Delta}(g_n^k)^{-1}\phi_n^l = (g_n^k)^{-1}g_n^j e^{i\left(t_n^j - t_n^k\frac{(\lambda_n^k)^2}{(\lambda_n^j)^2}\right)\Delta}[e^{-it_n^j\Delta}(g_n^j)^{-1}\phi_n^l].$$

Arguing as for the first term on RHS(4.9), it thus suffices to show that

$$e^{-it_n^j\Delta}(g_n^j)^{-1}\phi_n^l = e^{-it_n^j\Delta}(g_n^j)^{-1}g_n^l e^{it_n^l\Delta}\phi^l \rightharpoonup 0 \quad \text{weakly in } \dot H_x^1.$$

Using a density argument, this reduces to

$$I_n := e^{-it_n^j\Delta}(g_n^j)^{-1}g_n^l e^{it_n^l\Delta}\phi \rightharpoonup 0 \quad \text{weakly in } \dot H_x^1, \tag{4.10}$$

for all $\phi \in C_c^\infty(\mathbb{R}^d)$. Note that we can rewrite I_n as follows:

$$I_n = \left(\frac{\lambda_n^j}{\lambda_n^l}\right)^{\frac{d-2}{2}}\left[e^{i\left(t_n^l - t_n^j\left(\frac{\lambda_n^j}{\lambda_n^l}\right)^2\right)\Delta}\phi\right]\left(\frac{\lambda_n^j x + x_n^j - x_n^l}{\lambda_n^l}\right).$$

Recalling that (4.6) holds for the pair (j,l), we first prove (4.10) when the scaling parameters are not comparable, that is,

$$\lim_{n\to\infty} \frac{\lambda_n^j}{\lambda_n^l} + \frac{\lambda_n^l}{\lambda_n^j} = \infty. \tag{4.11}$$

By Cauchy–Schwarz,

$$\begin{aligned}\big|\langle I_n, \psi\rangle_{\dot H_x^1}\big| &\lesssim \min\Big\{\|\Delta I_n\|_{L_x^2}\|\psi\|_{L_x^2}, \|I_n\|_{L_x^2}\|\Delta\psi\|_{L_x^2}\Big\}\\ &\lesssim \min\bigg\{\frac{\lambda_n^j}{\lambda_n^l}\|\Delta\phi\|_{L_x^2}\|\psi\|_{L_x^2}, \frac{\lambda_n^l}{\lambda_n^j}\|\phi\|_{L_x^2}\|\Delta\psi\|_{L_x^2}\bigg\},\end{aligned}$$

which converges to zero as $n\to\infty$, for all $\psi\in C_c^\infty(\mathbb{R}^d)$. This establishes (4.10) when (4.11) holds.

Henceforth we may assume

$$\lim_{n\to\infty}\frac{\lambda_n^j}{\lambda_n^l} = \lambda_1 \in (0,\infty).$$

We now suppose the time parameters diverge, that is,

$$\lim_{n\to\infty}\frac{|t_n^j(\lambda_n^j)^2 - t_n^l(\lambda_n^l)^2|}{\lambda_n^j\lambda_n^l} = \infty;$$

then we also have

$$\left|t_n^l - t_n^j\left(\frac{\lambda_n^j}{\lambda_n^l}\right)^2\right| = \frac{|t_n^l(\lambda_n^l)^2 - t_n^j(\lambda_n^j)^2|}{\lambda_n^l\lambda_n^j}\cdot\frac{\lambda_n^j}{\lambda_n^l}\to\infty \quad \text{as} \quad n\to\infty.$$

Under this condition, (4.10) follows from

$$\lambda_1^{\frac{d-2}{2}}\bigg[e^{i\big(t_n^l - t_n^j\big(\frac{\lambda_n^j}{\lambda_n^l}\big)^2\big)\Delta}\phi\bigg]\bigg(\lambda_1 x + \frac{x_n^j - x_n^l}{\lambda_n^l}\bigg)\rightharpoonup 0 \quad \text{weakly in } \dot H_x^1,$$

which is an immediate consequence of Exercise 4.3.

Finally, we deal with the situation when

$$\frac{\lambda_n^j}{\lambda_n^l}\to\lambda_1\in(0,\infty), \quad \frac{t_n^l(\lambda_n^l)^2 - t_n^j(\lambda_n^j)^2}{\lambda_n^j\lambda_n^l}\to t_1, \quad \text{but} \quad \frac{|x_n^j - x_n^l|^2}{\lambda_n^j\lambda_n^l}\to\infty. \tag{4.12}$$

Then we also have $t_n^l - t_n^j(\lambda_n^j)^2/(\lambda_n^l)^2 \to \lambda_1 t_1$. Thus, it suffices to show that

$$\lambda_1^{\frac{d-2}{2}} e^{it_1\lambda_1\Delta}\phi(\lambda_1 x + y_n)\rightharpoonup 0 \quad \text{weakly in } \dot H_x^1, \tag{4.13}$$

where

$$y_n := \frac{x_n^j - x_n^l}{\lambda_n^l} = \frac{x_n^j - x_n^l}{\sqrt{\lambda_n^l\lambda_n^j}}\sqrt{\frac{\lambda_n^j}{\lambda_n^l}}\to\infty \quad \text{as} \quad n\to\infty.$$

The desired weak convergence (4.13) follows again from Exercise 4.3.

Finally, we prove the last assertion in the theorem regarding the behaviour of t_n^j. For each j, by passing to a subsequence we may assume $t_n^j\to t^j\in[-\infty,\infty]$. Using a standard diagonal argument, we may assume that the limit exists for all $j\geq 1$.

Fix $j\geq 1$. If $t^j = \pm\infty$, there is nothing more to be proved. If $t^j\in(-\infty,\infty)$, we claim that we may redefine $t_n^j\equiv 0$, provided we replace the original profile ϕ^j

by $e^{it^j\Delta}\phi^j$. Indeed, we merely need to prove that the errors introduced by these changes can be incorporated into w_n^J, namely,

$$\lim_{n\to\infty}\big\|g_n^j e^{it_n^j\Delta}\phi^j - g_n^j e^{it^j\Delta}\phi^j\big\|_{\dot H_x^1} = 0.$$

But this follows easily from the strong convergence of the linear propagator.

This completes the proof of Theorem 4.1. □

Exercise 4.1. Under the hypotheses of Proposition 3.2, prove that

$$e^{-it_n^j\Delta}\big[(\lambda_n^j)^{\frac{d-2}{2}} w_n^J(\lambda_n^j x + x_n^j)\big] \rightharpoonup 0 \quad \text{weakly in } \dot H^1(\mathbb{R}^d) \text{ for all } j \le J.$$

Exercise 4.2. Let $f_n \in \dot H^1(\mathbb{R}^d)$ be such that $f_n \rightharpoonup 0$ weakly in $\dot H^1(\mathbb{R}^d)$ and let $t_n \to t_\infty \in \mathbb{R}$. Then

$$e^{it_n\Delta} f_n \rightharpoonup 0 \quad \text{weakly in } \dot H_x^1 \text{ as } n \to \infty.$$

Exercise 4.3. Let $f \in C_c^\infty(\mathbb{R}^d)$ and let $\{(t_n, x_n)\}_{n\ge1} \subset \mathbb{R}\times\mathbb{R}^d$. Then

$$e^{it_n\Delta} f(x + x_n) \rightharpoonup 0 \quad \text{weakly in } \dot H_x^1 \text{ as } n \to \infty$$

whenever $|t_n| \to \infty$ or $|x_n| \to \infty$.

5. Stability theory for the energy-critical NLS

In this section we develop a stability theory for the energy-critical NLS

$$i\partial_t u = -\Delta u \pm |u|^{\frac{4}{d-2}} u \quad \text{with} \quad u(0) = u_0 \in \dot H_x^1. \tag{5.1}$$

Definition 5.1 (Solution)**.** A function $u : I\times\mathbb{R}^d \to \mathbb{C}$ on a non-empty time interval $0 \in I \subset \mathbb{R}$ is a *solution* (more precisely, a strong $\dot H_x^1$ solution) to (5.1) if it lies in the class $C_t^0 \dot H_x^1(K\times\mathbb{R}^d) \cap L_{t,x}^{\frac{2(d+2)}{d-2}}(K\times\mathbb{R}^d)$ for all compact $K \subset I$, and satisfies the Duhamel formula

$$u(t) = e^{it\Delta}u(0) \mp i\int_0^t e^{i(t-s)\Delta}|u(s)|^{\frac{4}{d-2}}u(s)\,ds \tag{5.2}$$

for all $t \in I$. We refer to the interval I as the *lifespan* of u. We say that u is a *maximal-lifespan solution* if the solution cannot be extended to any strictly larger interval. We say that u is a *global solution* if $I = \mathbb{R}$.

Solutions to (5.1) conserve the energy

$$E(u(t)) = \int_{\mathbb{R}^d} \tfrac12|\nabla u(t,x)|^2 \pm \tfrac{d-2}{2d}|u(t,x)|^{\frac{2d}{d-2}}\,dx.$$

Note that taking data in $\dot H_x^1$ renders the energy finite. Indeed, Sobolev embedding shows that $\dot H_x^1$ is precisely the energy space.

The equation is called energy-critical because the scaling associated with this equation, namely,

$$u(t,x) \mapsto \lambda^{\frac{d-2}{2}} u\big(\lambda^2 t, \lambda x\big) \quad \text{for} \quad \lambda > 0,$$

leaves invariant not only the class of solutions to (5.1), but also the energy.

Throughout the section, we use S^0 to denote the intersection of any finite number of Strichartz spaces $L_t^q L_x^r$ with (q,r) obeying the conditions of Lemma 2.7, and N^0

to denote the sum of any finite number of dual Strichartz spaces $L_t^{q'} L_x^{r'}$. For an interval $I \subset \mathbb{R}$ we define the norms

$$\|u\|_{S^0(I)} := \|u\|_{S^0(I \times \mathbb{R}^d)} \quad \text{and} \quad \|F\|_{N^0(I)} := \|F\|_{N^0(I \times \mathbb{R}^d)}.$$

We start by reviewing the standard local well-posedness statement for (6.1).

Theorem 5.2 (Standard local well-posedness, [8, 9, 10])**.** *Let $d \geq 3$ and $u_0 \in H^1(\mathbb{R}^d)$. There exists $\eta_0 = \eta_0(d) > 0$ such that if $0 < \eta \leq \eta_0$ and I is a compact interval containing zero such that*

$$\big\|\nabla e^{it\Delta} u_0\big\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}(I \times \mathbb{R}^d)} \leq \eta, \tag{5.3}$$

then there exists a unique solution u to (5.1) on $I \times \mathbb{R}^d$. Moreover, we have the bounds

$$\begin{aligned} \big\|\nabla u\big\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}(I \times \mathbb{R}^d)} &\leq 2\eta \\ \big\|\nabla u\big\|_{S^0(I \times \mathbb{R}^d)} &\lesssim \big\|\nabla u_0\big\|_{L_x^2} + \eta^{1+p} \\ \|u\|_{S^0(I \times \mathbb{R}^d)} &\lesssim \|u_0\|_{L_x^2}. \end{aligned}$$

Proof. Exercise! *Hint:* use contraction mapping with the distance given by an S^0 norm. □

Remarks. 1. By the Strichartz inequality,

$$\big\|\nabla e^{it\Delta} u_0\big\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}(I \times \mathbb{R}^d)} \lesssim \big\|\nabla u_0\big\|_{L_x^2}.$$

Thus, (5.3) holds with $I = \mathbb{R}$ for initial data with sufficiently small $\dot{H}_x^1$ norm. In particular, we obtain global well-posedness for initial data in H_x^1 that is small in $\dot{H}_x^1$.

2. By the monotone convergence theorem, given an arbitrary $u_0 \in \dot{H}_x^1$ we can choose a sufficiently small interval I to ensure that (5.3) holds. Note however that the length of I will depend upon u_0 and not merely its norm.

This standard local well-posedness result suffers from the fact that the initial data belongs to the inhomogeneous Sobolev space H_x^1, rather than the energy space $\dot{H}_x^1$; the stronger requirement $u_0 \in H_x^1$ is needed in the proof of Theorem 5.2 in order to prove that the solution map is a contraction. To remove this restriction, we need the following stability result:

Theorem 5.3 (Energy-critical stability result, [22, 34])**.** *Let I a compact time interval and let $\tilde{u}$ be an approximate solution to (5.1) on $I \times \mathbb{R}^d$ in the sense that*

$$i\tilde{u}_t = -\Delta \tilde{u} \pm |\tilde{u}|^{\frac{4}{d-2}} \tilde{u} + e$$

for some function e. Assume that

$$\|\tilde{u}\|_{L_t^\infty \dot{H}_x^1(I \times \mathbb{R}^d)} \leq E \tag{5.4}$$

$$\|\tilde{u}\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I \times \mathbb{R}^d)} \leq L \tag{5.5}$$

for some positive constants E and L. Let $t_0 \in I$ and $u_0 \in \dot{H}_x^1$ and assume the smallness conditions

$$\|u_0 - \tilde{u}(t_0)\|_{\dot{H}_x^1} \leq \varepsilon \tag{5.6}$$

$$\|\nabla e\|_{N^0(I)} \le \varepsilon \tag{5.7}$$

for some $0 < \varepsilon < \varepsilon_1 = \varepsilon_1(E, L)$. *Then there exists a unique strong solution* $u : I \times \mathbb{R}^d \mapsto \mathbb{C}$ *to* (5.1) *with initial data* u_0 *at time* $t = t_0$ *satisfying*

$$\|u - \tilde u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I\times\mathbb{R}^d)} \le C(E, L)\varepsilon^c \tag{5.8}$$

$$\|\nabla(u - \tilde u)\|_{S^0(I)} \le C(E, L) \tag{5.9}$$

$$\|\nabla u\|_{S^0(I)} \le C(E, L), \tag{5.10}$$

where $c = c(d) > 0$.

This stability result was first proved for $d = 3$ in the work of Colliander, Keel, Staffilani, Takaoka, and Tao [13] on the defocusing energy-critical NLS. For $d = 4$, it can be found in [30]. The same proof extends easily to dimensions $d = 5, 6$. To prove Theorem 5.3 in dimensions $d \ge 7$, new ideas are needed. To see why, let us consider the question of continuous dependence of the solution upon the initial data, which corresponds to taking $e = 0$ in Theorem 5.3. To make things as simple as possible, we choose initial data $u_0, \tilde u_0 \in H_x^1$ which are small in the sense that

$$\|u_0\|_{\dot H_x^1} + \|\tilde u_0\|_{\dot H_x^1} \le \eta_0.$$

By the first remark above, if η_0 is sufficiently small there exist unique global solutions u and $\tilde u$ to (5.1) with initial data u_0 and $\tilde u_0$, respectively; moreover, they satisfy

$$\|\nabla u\|_{S^0(\mathbb{R})} + \|\nabla \tilde u\|_{S^0(\mathbb{R})} \lesssim \eta_0.$$

We would like to see that if u_0 and $\tilde u_0$ are *close* in $\dot H_x^1$, say $\|\nabla(u_0 - \tilde u_0)\|_2 \le \varepsilon \ll \eta_0$, then u and $\tilde u$ remain ε-close in *energy-critical* norms. An application of the Strichartz inequality combined with the bounds above yields

$$\|\nabla(u - \tilde u)\|_{S^0(\mathbb{R})} \lesssim \|\nabla(u_0 - \tilde u_0)\|_{L_x^2} + \eta_0^{\frac{4}{d-2}}\|\nabla(u - \tilde u)\|_{S^0(\mathbb{R})} + \eta_0\|\nabla(u - \tilde u)\|_{S^0(\mathbb{R})}^{\frac{4}{d-2}}.$$

If $4/(d-2) \ge 1$, a simple bootstrap argument implies continuous dependence of the solution upon the initial data. However, this will not work if $4/(d-2) < 1$, that is, if $d \ge 7$. The last term in the inequality above causes the bootstrap argument to break down in high dimensions; indeed, tiny numbers become much larger when raised to a fractional power.

To prove Theorem 5.3 in dimensions $d \ge 7$, the authors of [34] work in spaces with fractional derivatives (rather than a full derivative), while still maintaining criticality with respect to the scaling. A similar technique was employed by Nakanishi [27] for the energy-critical Klein–Gordon equation in high dimensions.

The result in [34] assumes the less stringent smallness condition

$$\Big(\sum_{N\in 2^{\mathbb{Z}}} \big\|\nabla P_N e^{i(t-t_0)\Delta}\big(u_0 - \tilde u(t_0)\big)\big\|^2_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}(I\times\mathbb{R}^d)}\Big)^{1/2} \le \varepsilon$$

in place of (5.6). There is also an improvement over the result in [34], in which the smallness condition above is replaced by

$$\big\|e^{i(t-t_0)\Delta}\big(u_0 - \tilde u(t_0)\big)\big\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I\times\mathbb{R}^d)} \le \varepsilon.$$

To prove Theorem 5.3 with this particular hypothesis (which was helpful in early treatments of the energy-critical NLS), it becomes necessary to work in spaces with fractional derivatives even in small dimensions; see [22] for the proof.

In what follows, we will present the proof of Theorem 5.3 in dimensions $3 \leq d \leq 6$. For higher dimensions, see [22, 34].

Proof of Theorem 5.3 for $3 \leq d \leq 6$. We will prove the result under the additional assumption that $u_0 \in L_x^2$ (and so $u_0 \in H_x^1$). This allows us to invoke Theorem 5.2 and so guarantee that u exists. Thus, it suffices to prove (5.8) through (5.10) as *a priori* estimates, that is, we assume that u exists and satisfies $\nabla u \in S^0(I)$. Once we have proved (5.8) through (5.10), we may remove the additional assumption $u_0 \in L_x^2$ by the usual limiting argument: Approximate $u_0 \in \dot H_x^1$ by $\{f_n\}_{n\geq 1} \subset H_x^1$ and let u_n be the solution to (5.1) with initial data $u_n(t_0) = f_n$. Applying Theorem 5.3 with $\tilde u := u_m$, $u := u_n$, and $e = 0$, we deduce that the sequence of solutions $\{u_n\}_{n\geq 1}$ is Cauchy in energy-critical norms. Therefore, u_n converges to a solution u with data $u(t_0) = u_0$ which satisfies $\nabla u \in S^0(I)$.

We first prove the theorem under the hypothesis

$$\|\nabla \tilde u\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}(I\times\mathbb{R}^d)} \leq \delta \tag{5.11}$$

for some $\delta > 0$ sufficiently small depending on E. Without loss of generality, we may assume $t_0 = \inf I$.

To continue, let $v := u - \tilde u$ and for $t \in I$ define

$$A(t) := \big\|\nabla\big[(i\partial_t + \Delta)v + e\big]\big\|_{L_t^2 L_x^{\frac{2d}{d+2}}([t_0,t]\times\mathbb{R}^d)}.$$

By Sobolev embedding, Strichartz, (5.6), and (5.7), we get

$$\begin{aligned}
\|v\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}([t_0,t]\times\mathbb{R}^d)} &\lesssim \|\nabla v\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}([t_0,t]\times\mathbb{R}^d)} \\
&\lesssim \|v(t_0)\|_{\dot H_x^1} + A(t) + \|\nabla e\|_{L_t^2 L_x^{\frac{2d}{d+2}}([t_0,t]\times\mathbb{R}^d)} \\
&\lesssim A(t) + \varepsilon. \qquad (5.12)
\end{aligned}$$

On the other hand, by Hölder, (5.11), (5.12), and Sobolev embedding, we get

$$\begin{aligned}
A(t) &\lesssim \|\nabla \tilde u\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}} \|v\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \Big[\|v\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} + \|\tilde u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}\Big]^{\frac{6-d}{d-2}} \\
&\quad + \|\nabla v\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}} \Big[\|v\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} + \|\tilde u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}\Big]^{\frac{4}{d-2}} \\
&\lesssim \delta[A(t)+\varepsilon][A(t)+\varepsilon+\delta]^{\frac{6-d}{d-2}} + [A(t)+\varepsilon][A(t)+\varepsilon+\delta]^{\frac{4}{d-2}},
\end{aligned}$$

where all spacetime norms are over $[t_0, t] \times \mathbb{R}^d$.

Taking δ, ε sufficiently small (depending only on the ambient dimension so far), a standard continuity argument gives

$$A(t) \lesssim \varepsilon \quad \text{for all} \quad t \in I \tag{5.13}$$

with $c = c(d) = 1$. Together with (5.12), this gives (5.8). To obtain (5.9), we use the Strichartz inequality, (5.6), (5.7), and (5.13), as follows:

$$\begin{aligned}
\|\nabla(u-\tilde u)\|_{S^0(I)} &\lesssim \|u_0 - \tilde u(t_0)\|_{\dot H_x^1} + \big\|\nabla\big[(i\partial_t+\Delta)v + e\big]\big\|_{L_t^2 L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)} \\
&\quad + \|\nabla e\|_{L_t^2 L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)} \\
&\lesssim \varepsilon.
\end{aligned}$$

To obtain (5.10), we first note that by (5.11) and (5.12),

$$\begin{aligned}\|\nabla u\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}(I\times\mathbb{R}^d)} &\leq \|\nabla v\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}(I\times\mathbb{R}^d)} + \|\nabla \tilde u\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}(I\times\mathbb{R}^d)} \\ &\lesssim \varepsilon + \delta.\end{aligned}$$

Using this together with the Strichartz inequality, Sobolev embedding, and (5.4),

$$\begin{aligned}\|\nabla u\|_{S^0(I)} &\lesssim \|\tilde u(t_0)\|_{\dot H^1_x} + \|u_0 - \tilde u(t_0)\|_{\dot H^1_x} + \|\nabla u\|^{\frac{d+2}{d-2}}_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}(I\times\mathbb{R}^d)} \\ &\lesssim E + \varepsilon + [\varepsilon+\delta]^{\frac{d+2}{d-2}} \lesssim E,\end{aligned}$$

provided $\delta, \varepsilon \leq \varepsilon_0 = \varepsilon_0(E)$.

To complete the proof of Theorem 5.3 in small dimensions, it remains to restore the hypothesis (5.5) in place of (5.11). We first note that (5.5) implies $\nabla u \in S^0(I)$. Indeed, subdividing I into $N_0 \sim (1+\frac{L}{\eta})^{\frac{2(d+2)}{d-2}}$ subintervals J_k such that on each J_k we have

$$\|\tilde u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(J_k\times\mathbb{R}^d)} \leq \eta,$$

and using the Strichartz inequality, Sobolev embedding, and (5.4), we estimate

$$\begin{aligned}\|\nabla \tilde u\|_{S^0(J_k)} &\lesssim \|\tilde u\|_{L_t^\infty \dot H^1_x(I\times\mathbb{R}^d)} + \|\nabla \tilde u\|_{S^0(J_k)} \|\tilde u\|^{\frac{4}{d-2}}_{L_{t,x}^{\frac{2(d+2)}{d-2}}(J_k\times\mathbb{R}^d)} + \|\nabla e\|_{L_t^2 L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)} \\ &\lesssim E + \eta^{\frac{4}{d-2}} \|\nabla \tilde u\|_{S^0(J_k)} + \varepsilon.\end{aligned}$$

Thus for η sufficiently small depending on d,

$$\|\nabla \tilde u\|_{S^0(J_k)} \lesssim E + \varepsilon.$$

Summing these bounds over all the intervals J_k we obtain

$$\|\nabla \tilde u\|_{S^0(I)} \leq C(E, L).$$

We can now subdivide I into $N_1 = N_1(E, L)$ subintervals $I_j = [t_j, t_{j+1}]$ such that on each I_j we have

$$\|\nabla \tilde u\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}(I_j\times\mathbb{R}^d)} \leq \delta,$$

where δ is as in (5.11). Choosing ε_1 sufficiently small depending on ε_0 and N_1, the argument above implies that for each j and all $0 < \varepsilon < \varepsilon_1$,

$$\begin{aligned}\|u - \tilde u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I_j\times\mathbb{R}^d)} &\leq C(j)\varepsilon \\ \|\nabla(u-\tilde u)\|_{S^0(I_j)} &\leq C(j)\varepsilon \\ \|\nabla u\|_{S^0(I_j)} &\leq C(j)E \\ \big\|\nabla\big[(i\partial_t+\Delta)(u-\tilde u)+e\big]\big\|_{L_t^2 L_x^{\frac{2d}{d+2}}(I_j\times\mathbb{R}^d)} &\leq C(j)\varepsilon,\end{aligned}$$

provided we can show that (5.6) holds when t_0 is replaced by t_j. We check this using an inductive argument. By the Strichartz inequality,

$$\begin{aligned}\|u(t_{j+1}) - \tilde u(t_{j+1})\|_{\dot H^1_x} &\lesssim \|u_0 - \tilde u(t_0)\|_{\dot H^1_x} + \|\nabla e\|_{L_t^2 L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)} \\ &\quad + \big\|\nabla\big[(i\partial_t+\Delta)(u-\tilde u)+e\big]\big\|_{L_t^2 L_x^{\frac{2d}{d+2}}([t_0,t_{j+1}]\times\mathbb{R}^d)}\end{aligned}$$

$$\lesssim \varepsilon + \sum_{k=0}^{j} C(k)\varepsilon.$$

Choosing ε_1 sufficiently small depending on ε_0 and E, we can continue the inductive argument.

This completes the proof of Theorem 5.3 in dimensions $3 \leq d \leq 6$. $\square$

6. A large data critical problem

Throughout the remainder of these notes we restrict attention to the defocusing energy-critical NLS

$$i\partial_t u + \Delta u = |u|^{\frac{4}{d-2}} u \quad \text{with} \quad u(0) = u_0 \in \dot{H}^1_x. \tag{6.1}$$

For arguments and further references in the focusing case, see [22]. For equation (6.1) we have the following large data global result:

Theorem 6.1 (Global well-posedness and scattering)**.** *Let $d \geq 3$ and $u_0 \in \dot{H}^1_x$. Then there exists a unique global solution u to* (6.1) *and it satisfies*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+2)}{d-2}}\, dx\, dt \leq C(E(u_0)).$$

In particular, the solution scatters, that is, there exist asymptotic states $u_\pm \in \dot{H}^1_x$ such that

$$\|u(t) - e^{it\Delta} u_\pm\|_{\dot{H}^1_x} \to 0 \quad \textit{as} \quad t \to \pm\infty.$$

The proof of this theorem sparked the recent progress on dispersive equations at the critical regularity. It was first proved for spherically symmetric initial data in dimensions $d = 3, 4$ by Bourgain [5]. In this work, Bourgain introduced the induction on energy paradigm as a means for breaking the scaling symmetry; this allowed him to use non-critical monotonicity formulas like the Morawetz inequality (which scales like $\dot{H}^{1/2}_x$). Building on Bourgain's argument, Tao [33] proved the theorem in dimensions $d \geq 5$ for spherically symmetric data.

The radial assumption was removed in dimension $d = 3$ by Colliander, Keel, Staffilani, Takaoka, and Tao [13]. This work further advanced the induction on energy argument, introducing important new ideas that informed subsequent developments. To deal with arbitrary data, the authors employed a frequency-localized interaction Morawetz inequality, which is even further away from scaling (it scales like $\dot{H}^{1/4}_x$). The work [13] was extended to four dimensions in [30]. Finally, for dimensions $d \geq 5$, Theorem 6.1 was proved in [39]; for a different proof reflecting new advances see [23], which also treats the focusing problem.

In these notes, we will present the proof of Theorem 6.1 in dimension $d = 4$. The proof below is taken from [40], which revisits the argument in [30] in light of the recent advances made by Dodson [15] on the mass-critical NLS. For a proof of the three-dimensional case treated in [13] that also incorporates these advances see [21].

We note that parts of the argument we will present in these notes work in all dimensions $d \geq 3$; in particular, we will demonstrate the existence of a minimal counterexample to Theorem 6.1 in all dimensions $d \geq 3$.

To start, for any $0 \leq E < \infty$, we define

$$L(E) := \sup\{S_I(u) : \, u : I \times \mathbb{R}^d \to \mathbb{C} \text{ such that } E(u) \leq E\},$$

where the supremum is taken over all solutions $u : I \times \mathbb{R}^d \to \mathbb{C}$ to (6.1). Here, we use the notation

$$S_I(u) := \int_I \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+2)}{d-2}}\, dx\, dt$$

for the scattering size of u on an interval I.

Note that $L : [0,\infty) \to [0,\infty]$ is a non-decreasing function. Moreover, from the small data theory,

$$L(E) \lesssim E^{\frac{d+2}{d-2}} \quad \text{for} \quad E \leq \eta_0,$$

where $\eta_0 = \eta_0(d)$ is the small data threshold.

Exercise 6.1. Prove that the set $\{E > 0 : L(E) < \infty\}$ is open.
Hint: Use Theorem 5.3.

Therefore, there must exist a unique *critical energy* $0 < E_c \leq \infty$ such that

$$L(E) < \infty \quad \text{for} \quad E < E_c \quad \text{and} \quad L(E) = \infty \quad \text{for} \quad E \geq E_c.$$

This plays the role of the inductive hypothesis because it says that Theorem 6.1 holds for energies $E < E_c$. The argument is called induction on energy, because this inductive hypothesis will be used to prove that $L(E_c) < \infty$, thus providing the desired contradiction.

7. A Palais–Smale type condition

In this section we prove a Palais–Smale condition for minimizing sequences of blowup solutions to the defocusing energy-critical NLS. It was already observed in [5, 13] that such minimizing sequences have good tightness and equicontinuity properties. This was taken to its ultimate conclusion by Keraani [19] who showed the existence and almost periodicity of minimal blowup solutions in the context of the mass-critical NLS. The proof of the Palais–Smale condition is the crux of this argument.

We first define operators T_n^j on general functions of spacetime. These act on linear solutions in a manner corresponding to the action of $g_n^j e^{it_n^j\Delta}$ on initial data:

$$(T_n^j u)(t,x) := (\lambda_n^j)^{-\frac{d-2}{2}} u\biggl(\frac{t}{(\lambda_n^j)^2} + t_n^j, \frac{x - x_n^j}{\lambda_n^j}\biggr).$$

Here, the parameters $\lambda_n^j, t_n^j, x_n^j$ are as defined in Theorem 4.1. Using the asymptotic orthogonality condition (4.6), it is not hard to prove the following

Lemma 7.1 (Asymptotic decoupling)**.** *Suppose that the parameters associated to j,k are orthogonal in the sense of (4.6). Then for any $\psi^j, \psi^k \in C_c^\infty(\mathbb{R}\times\mathbb{R}^d)$,*

$$\|T_n^j\psi^j T_n^k\psi^k\|_{L_{t,x}^{\frac{d+2}{d-2}}} + \|T_n^j\psi^j \nabla(T_n^k\psi^k)\|_{L_{t,x}^{\frac{d+2}{d-1}}} + \|\nabla(T_n^j\psi^j)\nabla(T_n^k\psi^k)\|_{L_{t,x}^{\frac{d+2}{d}}}$$

converges to zero as $n \to \infty$.

Proof. From a change of variables, we get

$$\begin{aligned} &\|T_n^j\psi^j T_n^k\psi^k\|_{L_{t,x}^{\frac{d+2}{d-2}}} + \|T_n^j\psi^j \nabla(T_n^k\psi^k)\|_{L_{t,x}^{\frac{d+2}{d-1}}} + \|\nabla(T_n^j\psi^j)\nabla(T_n^k\psi^k)\|_{L_{t,x}^{\frac{d+2}{d}}} \\ &\quad = \|\psi^j (T_n^j)^{-1}T_n^k\psi^k\|_{L_{t,x}^{\frac{d+2}{d-2}}} + \|\psi^j \nabla(T_n^j)^{-1}T_n^k\psi^k\|_{L_{t,x}^{\frac{d+2}{d-1}}} + \|\nabla\psi^j \nabla(T_n^j)^{-1}T_n^k\psi^k\|_{L_{t,x}^{\frac{d+2}{d}}}, \end{aligned}$$

where all spacetime norms are over $\mathbb{R}\times\mathbb{R}^d$. Note that

$$[(T_n^j)^{-1}T_n^k\psi^k](t,x)=\big(\tfrac{\lambda_n^j}{\lambda_n^k}\big)^{\frac{d-2}{2}}\psi^k\Big(\big(\tfrac{\lambda_n^j}{\lambda_n^k}\big)^2\big(t-\tfrac{t_n^j(\lambda_n^j)^2-t_n^k(\lambda_n^k)^2}{(\lambda_n^j)^2}\big),\tfrac{\lambda_n^j}{\lambda_n^k}\big(x-\tfrac{x_n^k-x_n^j}{\lambda_n^j}\big)\Big).$$

We will only present the details for decoupling in the $L_{t,x}^{\frac{d+2}{d-2}}$ norm; the argument for decoupling in the other norms is very similar.

We first assume that $\frac{\lambda_n^j}{\lambda_n^k}+\frac{\lambda_n^k}{\lambda_n^j}\to\infty$. Using Hölder's inequality and a change of variables, we estimate

$$\begin{aligned}
&\|\psi^j(T_n^j)^{-1}T_n^k\psi^k\|_{L_{t,x}^{\frac{d+2}{d-2}}}\\
&\qquad\le\min\big\{\|\psi^j\|_{L^\infty_{t,x}}\|(T_n^j)^{-1}T_n^k\psi^k\|_{L_{t,x}^{\frac{d+2}{d-2}}}+\|\psi^j\|_{L_{t,x}^{\frac{d+2}{d-2}}}\|(T_n^j)^{-1}T_n^k\psi^k\|_{L^\infty_{t,x}}\big\}\\
&\qquad\lesssim\min\Big\{\big(\tfrac{\lambda_n^j}{\lambda_n^k}\big)^{-\frac{d-2}{2}},\big(\tfrac{\lambda_n^j}{\lambda_n^k}\big)^{\frac{d-2}{2}}\Big\}\to0\quad\text{as}\quad n\to\infty.
\end{aligned}$$

Henceforth, we may assume $\frac{\lambda_n^j}{\lambda_n^k}\to\lambda_0\in(0,\infty)$.

If $\frac{|t_n^j(\lambda_n^j)^2-t_n^k(\lambda_n^k)^2|}{\lambda_n^k\lambda_n^j}\to\infty$, it is easy to see that the temporal supports of ψ^j and $(T_n^j)^{-1}T_n^k\psi^k$ become disjoint for n sufficiently large. Hence

$$\lim_{n\to\infty}\|\psi^j(T_n^j)^{-1}T_n^k\psi^k\|_{L_{t,x}^{\frac{d+2}{d-2}}}=0.$$

If instead

$$\frac{\lambda_n^j}{\lambda_n^k}\to\lambda_0,\qquad\frac{t_n^j(\lambda_n^j)^2-t_n^k(\lambda_n^k)^2}{\lambda_n^k\lambda_n^j}\to t_0,\quad\text{and}\quad\frac{|x_n^j-x_n^k|}{\sqrt{\lambda_n^j\lambda_n^k}}\to\infty,$$

then the spatial supports of ψ^j and $(T_n^j)^{-1}T_n^k\psi^k$ become disjoint for n sufficiently large. Indeed, in this case we have

$$\frac{|x_n^j-x_n^k|}{\lambda_n^j}=\frac{|x_n^j-x_n^k|}{\sqrt{\lambda_n^j\lambda_n^k}}\sqrt{\frac{\lambda_n^k}{\lambda_n^j}}\to\infty\quad\text{as}\quad n\to\infty.$$

This completes the proof of the lemma. □

Recall that failure of Theorem 6.1 implies the existence of a critical energy $0<E_c<\infty$ so that

$$L(E)<\infty\quad\text{for}\quad E<E_c\quad\text{and}\quad L(E)=\infty\quad\text{for}\quad E\ge E_c,\tag{7.1}$$

where $L(E)$ denotes the supremum of $S_I(u)$ over all solutions $u:I\times\mathbb{R}^d\to\mathbb{C}$ with $E(u)\le E$.

The positivity of E_c is a consequence of the small data global well-posedness. Indeed, the argument proves the stronger statement

$$\|u\|_{\dot X^1(\mathbb{R}\times\mathbb{R}^d)}\lesssim E(u_0)^{\frac12}\quad\text{for all data with } E(u_0)\le\eta_0,\tag{7.2}$$

where η_0 denotes the small data threshold. Here,

$$\dot X^1:=L_{t,x}^{\frac{2(d+2)}{d-2}}\cap L_t^{\frac{2(d+2)}{d}}\dot H_x^{1,\frac{2(d+2)}{d}}.$$

Using the induction on energy argument together with (7.1) and the stability result Theorem 5.3, we now prove a compactness result for optimizing sequences of blowup solutions.

Proposition 7.2 (Palais–Smale condition)**.** *Let $u_n : I_n \times \mathbb{R}^d \to \mathbb{C}$ be a sequence of solutions to the defocusing energy-critical NLS with $E(u_n) \to E_c$, for which there is a sequence of times $t_n \in I_n$ so that*

$$\lim_{n\to\infty} S_{\geq t_n}(u_n) = \lim_{n\to\infty} S_{\leq t_n}(u_n) = \infty.$$

Then the sequence $u_n(t_n)$ has a subsequence that converges in $\dot H^1_x$ modulo scaling and spatial translations.

Proof. Using time translation symmetry, we may take $t_n \equiv 0$ for all n; thus,

$$\lim_{n\to\infty} S_{\geq 0}(u_n) = \lim_{n\to\infty} S_{\leq 0}(u_n) = \infty. \tag{7.3}$$

Applying Theorem 4.1 to the bounded sequence $\{u_n(0)\}_{n\geq 1} \subset \dot H^1_x$ and passing to a subsequence if necessary, we decompose

$$u_n(0) = \sum_{j=1}^{J} g_n^j e^{it_n^j \Delta}\phi^j + w_n^J \tag{7.4}$$

with the properties stated in that theorem. In particular, for any finite $0 \leq J \leq J^*$ we have the energy decoupling condition

$$\lim_{n\to\infty} \Big\{ E(u_n) - \sum_{j=1}^{J} E(e^{it_n^j\Delta}\phi^j) - E(w_n^J) \Big\} = 0. \tag{7.5}$$

To prove the proposition, we need to show that $J^* = 1$, that $w_n^1 \to 0$ in $\dot H^1_x$, and that $t_n^1 \equiv 0$. All other possibilities will be shown to contradict (7.3). We discuss two scenarios:

Scenario I: $\sup_j \limsup_{n\to\infty} E(e^{it_n^j\Delta}\phi^j) = E_c$.

From the non-triviality of the profiles, we have $\liminf_{n\to\infty} E(e^{it_n^j\Delta}\phi^j) > 0$ for every finite $1 \leq j \leq J^*$. Thus, using (7.5) together with the hypothesis $E(u_n) \to E_c$ (and passing to a subsequence if necessary), we deduce that there is a single profile in the decomposition (7.4) (that is, $J^* = 1$) and we can write

$$u_n(0) = g_n e^{it_n\Delta}\phi + w_n \quad \text{with} \quad \lim_{n\to\infty} \|w_n\|_{\dot H^1_x} = 0 \tag{7.6}$$

and $t_n \equiv 0$ or $t_n \to \pm\infty$. If $t_n \equiv 0$, then we obtain the desired compactness. Thus, we only need to preclude the scenario when $t_n \to \pm\infty$.

Let us suppose $t_n \to \infty$; the case $t_n \to -\infty$ can be treated symmetrically. In this case, the Strichartz inequality and the monotone convergence theorem yield

$$S_{\geq 0}(e^{it\Delta}u_n(0)) \lesssim S_{\geq t_n}(e^{it\Delta}\phi) + S(e^{it\Delta}w_n) \to 0 \quad \text{as} \quad n \to \infty.$$

By Theorem 5.3, this implies that $S_{\geq 0}(u_n) \to 0$, which contradicts (7.3).

Scenario 2: $\sup_j \limsup_{n\to\infty} E(e^{it_n^j\Delta}\phi^j) \leq E_c - 2\delta$ for some $\delta > 0$.

We first observe that in this case, for each finite $J \leq J^*$ we have $E(e^{it_n^j\Delta}\phi^j) \leq E_c - \delta$ for all $1 \leq j \leq J$ and n sufficiently large.

Next we define nonlinear profiles corresponding to each bubble in the decomposition of $u_n(0)$. If $t_n^j \equiv 0$, we define $v^j : I^j \times \mathbb{R}^d \to \mathbb{C}$ to be the maximal-lifespan solution to the defocusing energy-critical NLS with initial data $v^j(0) = \phi^j$. If instead $t_n^j \to \pm\infty$, we define $v^j : I^j \times \mathbb{R}^d \to \mathbb{C}$ to be the maximal-lifespan solution to the defocusing energy-critical NLS which scatters to $e^{it\Delta}\phi^j$ as $t \to \pm\infty$. Now define $v_n^j := T_n^j v^j$. Then v_n^j is also a solution to the defocusing energy-critical NLS on the time interval $I_n^j := (\lambda_n^j)^2(I^j - \{t_n^j\})$. In particular, for n sufficiently large we have $0 \in I_n^j$ and

$$\lim_{n\to\infty} \|v_n^j(0) - g_n^j e^{it_n^j\Delta}\phi^j\|_{\dot H_x^1} = 0. \tag{7.7}$$

Combining this with $E(e^{it_n^j\Delta}\phi^j) \le E_c - \delta < E_c$ and the inductive hypothesis (7.1), we deduce that for n sufficiently large, v_n^j (and so also v^j) are global solutions that satisfy

$$S_{\mathbb{R}}(v^j) = S_{\mathbb{R}}(v_n^j) \le L(E_c - \delta) < \infty.$$

(Note in particular that this implies v_n^j are global for *all* $n \ge 1$ and they admit a common spacetime bound.)

Combining this with the Strichartz inequality shows that all Strichartz norms of v^j and v_n^j are finite; in particular,

$$\|v^j\|_{\dot X^1(\mathbb{R}\times\mathbb{R}^d)} = \|v_n^j\|_{\dot X^1(\mathbb{R}\times\mathbb{R}^d)} \le_{E_c,\delta} 1.$$

This allows us to approximate v_n^j in $\dot X^1(\mathbb{R}\times\mathbb{R}^d)$ by $C_c^\infty(\mathbb{R}\times\mathbb{R}^d)$ functions. More precisely, for any $\varepsilon > 0$ there exist $\psi_\varepsilon^j \in C_c^\infty(\mathbb{R}\times\mathbb{R}^d)$ so that

$$\|v_n^j - T_n^j\psi_\varepsilon^j\|_{\dot X^1(\mathbb{R}\times\mathbb{R}^d)} < \varepsilon. \tag{7.8}$$

Moreover, we may use (7.2) together with our bounds on the spacetime norms of v_n^j and the finiteness of E_c to deduce that

$$\|v_n^j\|_{\dot X^1(\mathbb{R}\times\mathbb{R}^d)} \lesssim_{E_c,\delta} E(e^{it_n^j\Delta}\phi^j)^{\frac12} \lesssim_{E_c,\delta} 1. \tag{7.9}$$

Combining this with (7.5) we deduce that

$$\limsup_{n\to\infty}\sum_{j=1}^J \|v_n^j\|^2_{\dot X^1(\mathbb{R}\times\mathbb{R}^d)} \lesssim_{E_c,\delta} \limsup_{n\to\infty}\sum_{j=1}^J E(e^{it_n^j\Delta}\phi_n^j) \lesssim_{E_c,\delta} 1, \tag{7.10}$$

uniformly for finite $J \le J^*$.

The asymptotic orthogonality condition (4.6) gives rise to asymptotic decoupling of the nonlinear profiles.

Lemma 7.3 (Decoupling of nonlinear profiles)**.** *For $j \ne k$ we have*

$$\lim_{n\to\infty} \|v_n^j v_n^k\|_{L_{t,x}^{\frac{d+2}{d-2}}(\mathbb{R}\times\mathbb{R}^d)} + \|v_n^j \nabla v_n^k\|_{L_{t,x}^{\frac{d+2}{d-1}}(\mathbb{R}\times\mathbb{R}^d)} + \|\nabla v_n^j \nabla v_n^k\|_{L_{t,x}^{\frac{d+2}{d}}(\mathbb{R}\times\mathbb{R}^d)} = 0.$$

Proof. We only present the argument for decoupling in the $L_{t,x}^{\frac{d+2}{d-2}}$ norm; the argument for decoupling in the other norms is similar. Recall that for any $\varepsilon > 0$ there exist $\psi_\varepsilon^j, \psi_\varepsilon^k \in C_c^\infty(\mathbb{R}\times\mathbb{R}^d)$ so that

$$\|v_n^j - T_n^j\psi_\varepsilon^j\|_{\dot X^1(\mathbb{R}\times\mathbb{R}^d)} + \|v_n^k - T_n^k\psi_\varepsilon^k\|_{\dot X^1(\mathbb{R}\times\mathbb{R}^d)} < \varepsilon.$$

Thus, using (7.9) and Lemma 7.1 we get

$$\begin{aligned}
&\|v_n^j v_n^k\|_{L_{t,x}^{\frac{d+2}{d-2}}} \\
&\quad\le \|v_n^j(v_n^k - T_n^k\psi_\varepsilon^k)\|_{L_{t,x}^{\frac{d+2}{d-2}}} + \|(v_n^j - T_n^j\psi_\varepsilon^j)T_n^k\psi_\varepsilon^k\|_{L_{t,x}^{\frac{d+2}{d-2}}} + \|T_n^j\psi_\varepsilon^j\, T_n^k\psi_\varepsilon^k\|_{L_{t,x}^{\frac{d+2}{d-2}}} \\
&\quad\lesssim \|v_n^j\|_{\dot X^1}\|v_n^k - T_n^k\psi_\varepsilon^k\|_{\dot X^1} + \|v_n^j - T_n^j\psi_\varepsilon^j\|_{\dot X^1}\|\psi_\varepsilon^k\|_{\dot X^1} + \|T_n^j\psi_\varepsilon^j\, T_n^k\psi_\varepsilon^k\|_{L_{t,x}^{\frac{d+2}{d-2}}} \\
&\quad\lesssim_{E_c,\delta} \varepsilon + o(1) \quad \text{as} \quad n\to\infty.
\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, this proves the asymptotic decoupling statement. $\square$

As a consequence of this decoupling we can bound the sum of the nonlinear profiles in $\dot X^1$, as follows:

$$\limsup_{n\to\infty} \Big\| \sum_{j=1}^{J} v_n^j \Big\|_{\dot X^1(\mathbb{R}\times\mathbb{R}^d)} \lesssim_{E_c,\delta} 1 \quad \text{uniformly for finite } J \le J^*. \tag{7.11}$$

Indeed, by Young's inequality, (7.9), (7.10), and Lemma 7.3,

$$S_{\mathbb{R}}\Big(\sum_{j=1}^{J} v_n^j\Big) \lesssim \sum_{j=1}^{J} S_{\mathbb{R}}(v_n^j) + C_J \sum_{j\neq k} \|v_n^j v_n^k\|_{L_{t,x}^{\frac{d+2}{d-2}}}^{\frac{d+2}{d-2}} \lesssim_{E_c,\delta} 1 + C_J o(1) \quad \text{as} \quad n\to\infty.$$

Similarly,

$$\begin{aligned}\Big\| \sum_{j=1}^{J} \nabla v_n^j \Big\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^2 &= \Big\| \Big(\sum_{j=1}^{J} \nabla v_n^j\Big)^2 \Big\|_{L_{t,x}^{\frac{d+2}{d}}} \lesssim \sum_{j=1}^{J} \|\nabla v_n^j\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^2 + \sum_{j\neq k} \|\nabla v_n^j \nabla v_n^k\|_{L_{t,x}^{\frac{d+2}{d}}} \\ &\lesssim_{E_c,\delta} 1 + o(1) \quad \text{as} \quad n\to\infty.\end{aligned}$$

This completes the proof of (7.11). The same argument combined with (7.5) shows that given $\eta > 0$, there exists $J' = J'(\eta)$ such that

$$\limsup_{n\to\infty} \Big\| \sum_{j=J'}^{J} v_n^j \Big\|_{\dot X^1(\mathbb{R}\times\mathbb{R}^d)} \le \eta \quad \text{uniformly in } J \ge J'. \tag{7.12}$$

Now we are ready to construct an approximate solution to the defocusing energy-critical NLS. For each n and J, we define

$$u_n^J := \sum_{j=1}^{J} v_n^j + e^{it\Delta} w_n^J.$$

Obviously u_n^J is defined globally in time. In order to apply the stability result, it suffices to verify the following three claims for u_n^J:

Claim 1: $\|u_n^J(0) - u_n(0)\|_{\dot H_x^1} \to 0$ as $n\to\infty$ for any J.

Claim 2: $\limsup_{n\to\infty} \|u_n^J\|_{\dot X^1(\mathbb{R}\times\mathbb{R}^d)} \lesssim_{E_c,\delta} 1$ uniformly in J.

Claim 3: $\lim_{J\to J^*} \limsup_{n\to\infty} \big\| \nabla\big[(i\partial_t + \Delta) u_n^J - |u_n^J|^{\frac{4}{d-2}} u_n^J\big]\big\|_{N^0(\mathbb{R})} = 0$.

The three claims imply that for sufficiently large n and J, u_n^J is an approximate solution to the defocusing energy-critical NLS with finite scattering size, which asymptotically matches $u_n(0)$ at time $t=0$. Using the stability result we see that for n, J sufficiently large, the solution u_n inherits the spacetime bounds of u_n^J, thus contradicting (7.3). Therefore, to complete the treatment of the second scenario, it suffices to verify the three claims above.

The first claim follows trivially from (7.4) and (7.7). To derive the second claim, we use (7.11) and the Strichartz inequality, as follows:

$$\limsup_{n\to\infty} \|u_n^J\|_{\dot X^1(\mathbb{R}\times\mathbb{R}^d)} \lesssim \limsup_{n\to\infty} \Big\| \sum_{j=1}^{J} v_n^j \Big\|_{\dot X^1(\mathbb{R}\times\mathbb{R}^d)} + \limsup_{n\to\infty} \|w_n^J\|_{\dot H_x^1} \lesssim_{E_c,\delta} 1.$$

It remains to verify the third claim. Adopting the notation $F(z) = |z|^{\frac{4}{d-2}} z$, we write

$$
\begin{aligned}
(i\partial_t + \Delta) u_n^J - F(u_n^J) &= \sum_{j=1}^{J} F(v_n^j) - F(u_n^J) \\
&= \sum_{j=1}^{J} F(v_n^j) - F\Big(\sum_{j=1}^{J} v_n^j\Big) + F\big(u_n^J - e^{it\Delta} w_n^J\big) - F(u_n^J). \qquad (7.13)
\end{aligned}
$$

Taking the derivative, in dimensions $d \geq 6$ we estimate

$$
\Big|\nabla \Big\{ \sum_{j=1}^{J} F(v_n^j) - F\Big(\sum_{j=1}^{J} v_n^j\Big)\Big\}\Big| \lesssim_J \sum_{j \neq k} |\nabla v_n^j| |v_n^k|^{\frac{4}{d-2}}.
$$

In dimensions $d = 3, 4, 5$ there is an additional term on the right-hand side of the inequality above, namely, $\sum_{j \neq k} |\nabla v_n^j| |v_n^k| |v_n^j|^{\frac{6-d}{d-2}}$. Using (7.9) and Lemma 7.3, in dimensions $d \geq 6$ we estimate

$$
\begin{aligned}
\Big\|\nabla \Big[\sum_{j=1}^{J} F(v_n^j) - F\Big(\sum_{j=1}^{J} v_n^j\Big)\Big]\Big\|_{N^0(\mathbb{R})} &\lesssim_J \sum_{j \neq k} \big\| |\nabla v_n^j| |v_n^k|^{\frac{4}{d-2}} \big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \\
&\lesssim_J \sum_{j \neq k} \|\nabla v_n^j v_n^k\|_{L_{t,x}^{\frac{d+2}{d-1}}}^{\frac{4}{d-2}} \|\nabla v_n^k\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{d-6}{d-2}} \\
&\lesssim_{J, E_c, \delta} o(1) \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

The additional term in dimensions $d = 3, 4, 5$ can be treated analogously. Thus,

$$
\lim_{J \to J^*} \limsup_{n \to \infty} \Big\| \nabla \Big[\sum_{j=1}^{J} F(v_n^j) - F\Big(\sum_{j=1}^{J} v_n^j\Big)\Big]\Big\|_{N^0(\mathbb{R})} = 0. \qquad (7.14)
$$

We now turn to estimating the second difference in (7.13). We will show that

$$
\lim_{J \to J^*} \limsup_{n \to \infty} \big\| \nabla \big[F(u_n^J - e^{it\Delta} w_n^J) - F(u_n^J)\big]\big\|_{N^0(\mathbb{R})} = 0. \qquad (7.15)
$$

In dimensions $d \geq 6$,

$$
\begin{aligned}
\big\|\nabla \big[F\big(u_n^J - e^{it\Delta} w_n^J\big) - F(u_n^J)\big]\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} &\lesssim \|\nabla e^{it\Delta} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \|e^{it\Delta} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}} \\
&\quad + \|\nabla u_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \|e^{it\Delta} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}} \\
&\quad + \big\| |u_n^J|^{\frac{4}{d-2}} \nabla e^{it\Delta} w_n^J \big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}.
\end{aligned}
$$

In dimensions $d = 3, 4, 5$, one must add the term

$$
\|\nabla u_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \|e^{it\Delta} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \|u_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{6-d}{d-2}}
$$

to the right-hand side above. Using the second claim together with (4.2), and the Strichartz inequality combined with the fact that w_n^J is bounded in $\dot{H}_x^1$, we see that (7.15) will follow once we establish

$$
\lim_{J \to J^*} \limsup_{n \to \infty} \big\| |u_n^J|^{\frac{4}{d-2}} \nabla e^{it\Delta} w_n^J \big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(\mathbb{R} \times \mathbb{R}^d)} = 0. \qquad (7.16)
$$

We will only prove (7.16) in dimensions $d \geq 6$. We leave the remaining low dimensions as an exercise for the conscientious reader. Using Hölder's inequality, the second claim, and the Strichartz inequality, we get

$$\begin{aligned}
\big\| |u_n^J|^{\frac{4}{d-2}} \nabla e^{it\Delta} w_n^J \big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} &\lesssim \| u_n^J \nabla e^{it\Delta} w_n^J \|_{L_{t,x}^{\frac{d+2}{d-1}}}^{\frac{4}{d-2}} \| \nabla e^{it\Delta} w_n^J \|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{d-6}{d-2}} \\
&\lesssim_{E_c,\delta} \| e^{it\Delta} w_n^J \nabla e^{it\Delta} w_n^J \|_{L_{t,x}^{\frac{d+2}{d-1}}}^{\frac{4}{d-2}} + \Big\| \sum_{j=1}^J v_n^j \nabla e^{it\Delta} w_n^J \Big\|_{L_{t,x}^{\frac{d+2}{d-1}}}^{\frac{4}{d-2}} \\
&\lesssim_{E_c,\delta} \| e^{it\Delta} w_n^J \|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}} \| \nabla e^{it\Delta} w_n^J \|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{4}{d-2}} \\
&\qquad + \Big\| \sum_{j=1}^J v_n^j \nabla e^{it\Delta} w_n^J \Big\|_{L_{t,x}^{\frac{d+2}{d-1}}}^{\frac{4}{d-2}} \\
&\lesssim_{E_c,\delta} \| e^{it\Delta} w_n^J \|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}} + \Big\| \sum_{j=1}^J v_n^j \nabla e^{it\Delta} w_n^J \Big\|_{L_{t,x}^{\frac{d+2}{d-1}}}^{\frac{4}{d-2}}.
\end{aligned}$$

By (4.2), the contribution of the first term to (7.16) is acceptable. We now turn to the second term.

By (7.12),

$$\limsup_{n\to\infty} \Big\| \Big(\sum_{j=J'}^J v_n^j \Big) \nabla e^{it\Delta} w_n^J \Big\|_{L_{t,x}^{\frac{d+2}{d-1}}} \lesssim \limsup_{n\to\infty} \Big\| \sum_{j=J'}^J v_n^j \Big\|_{\dot X^1} \| \nabla e^{it\Delta} w_n^J \|_{L_{t,x}^{\frac{2(d+2)}{d}}} \lesssim_{E_c,\delta} \eta,$$

where $\eta > 0$ is arbitrary and $J' = J'(\eta)$ is as in (7.12). Thus, proving (7.16) reduces to showing

$$\lim_{J\to J^*} \limsup_{n\to\infty} \| v_n^j \nabla e^{it\Delta} w_n^J \|_{L_{t,x}^{\frac{d+2}{d-1}}} = 0 \quad \text{for each} \quad 1 \leq j < J'. \tag{7.17}$$

Fix $1 \leq j < J'$. By a change of variables,

$$\| v_n^j \nabla e^{it\Delta} w_n^J \|_{L_{t,x}^{\frac{d+2}{d-1}}} = \big\| v^j \nabla \tilde w_n^J \big\|_{L_{t,x}^{\frac{d+2}{d-1}}},$$

where $\tilde w_n^J := (T_n^j)^{-1}\big(e^{it\Delta} w_n^J\big)$. Note that

$$\| \tilde w_n^J \|_{\dot X^1(\mathbb{R}\times\mathbb{R}^d)} = \| e^{it\Delta} w_n^J \|_{\dot X^1(\mathbb{R}\times\mathbb{R}^d)}. \tag{7.18}$$

By density, we may assume $v^j \in C_c^\infty(\mathbb{R}\times\mathbb{R}^d)$. Invoking Hölder's inequality, it thus suffices to show

$$\lim_{J\to J^*} \limsup_{n\to\infty} \| \nabla \tilde w_n^J \|_{L^2_{t,x}(K)} = 0$$

for any compact $K \subset \mathbb{R}\times\mathbb{R}^d$. This however follows immediately from Lemma 2.12, (4.2), and (7.18), thus completing the proof of (7.17).

This proves (7.16) and so (7.15). Combining (7.14) and (7.15) yields the third claim. This completes the treatment of the second scenario and so the proof of the proposition. □

8. Existence of minimal blowup solutions and their properties

In this section we prove the existence of minimal counterexamples to Theorem 6.1 and we study some of their properties.

Theorem 8.1 (Existence of minimal counterexamples)**.** *Suppose Theorem 6.1 fails to be true. Then there exist a critical energy $0 < E_c < \infty$ and a maximal-lifespan solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ to the defocusing energy-critical NLS with $E(u) = E_c$, which blows up in both time directions in the sense that*

$$S_{\geq 0}(u) = S_{\leq 0}(u) = \infty,$$

and whose orbit $\{u(t) : t \in \mathbb{R}\}$ is precompact in $\dot{H}^1_x$ modulo scaling and spatial translations.

Proof. If Theorem 6.1 fails to be true, then there must exist a critical energy $0 < E_c < \infty$ and a sequence of solutions $u_n : I_n \times \mathbb{R}^d \to \mathbb{C}$ such that $E(u_n) \to E_c$ and $S_{I_n}(u_n) \to \infty$. Let $t_n \in I_n$ be such that $S_{\geq t_n}(u_n) = S_{\leq t_n}(u_n) = \frac{1}{2} S_{I_n}(u_n)$; then

$$\lim_{n\to\infty} S_{\geq t_n}(u_n) = \lim_{n\to\infty} S_{\leq t_n}(u_n) = \infty. \tag{8.1}$$

Applying Proposition 7.2 and passing to a subsequence, we find $\phi \in \dot{H}^1_x$ such that $u_n(t_n)$ converge to ϕ in $\dot{H}^1_x$ modulo scaling and spatial translations. Using the scaling and space-translation invariance of the equation and modifying $u_n(t_n)$ appropriately, we may assume $u_n(t_n) \to \phi$ in $\dot{H}^1_x$. In particular, $E(\phi) = E_c$.

Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be the maximal-lifespan solution to the defocusing energy-critical NLS with initial data $u(0) = \phi$. From the stability result Theorem 5.3 and (8.1), we get

$$S_{\geq 0}(u) = S_{\leq 0}(u) = \infty. \tag{8.2}$$

Finally, we prove that the orbit of u is precompact in $\dot{H}^1_x$ modulo scaling and space translations. For any sequence $\{t'_n\} \subset I$, (8.2) implies $S_{\geq t'_n}(u) = S_{\leq t'_n}(u) = \infty$. Thus by Proposition 7.2, we see that $u(t'_n)$ admits a subsequence that converges in $\dot{H}^1_x$ modulo scaling and space translations. This completes the proof of the theorem. ☐

By Corollary A.2, the maximal-lifespan solution found in Theorem 8.1 is *almost periodic modulo symmetries*, that is, there exist (possibly discontinuous) functions $N : I \to \mathbb{R}^+$, $x : I \to \mathbb{R}^d$, and $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\int_{|x-x(t)|\geq C(\eta)/N(t)} |\nabla u(t,x)|^2\,dx + \int_{|\xi|\geq C(\eta)N(t)} |\xi \hat{u}(t,\xi)|^2\,d\xi \leq \eta$$

for all $t \in I$ and $\eta > 0$. We refer to the function N as the *frequency scale function*, x is the *spatial center function*, and C is the *compactness modulus function.*

Another consequence of the precompactness in $\dot{H}^1_x$ modulo symmetries of the orbit of the solution found in Theorem 8.1 is that for every $\eta > 0$ there exists $c(\eta) > 0$ such that

$$\int_{|x-x(t)|\leq c(\eta)/N(t)} |\nabla u(t,x)|^2\,dx + \int_{|\xi|\leq c(\eta)N(t)} |\xi \hat{u}(t,\xi)|^2\,d\xi \leq \eta,$$

uniformly for all $t \in I$.

In what follows, we record some basic properties of almost periodic (modulo symmetries) solutions. We start with the following definition:

Definition 8.2 (Normalised solution)**.** Let $u : I \times \mathbb{R}^d \to C$ be a solution to (6.1), which is almost periodic modulo symmetries with parameters $N(t)$ and $x(t)$. We say that u is *normalised* if the lifespan I contains zero and

$$N(0) = 1 \quad \text{and} \quad x(0) = 0.$$

More generally, we can define the *normalisation* of a solution u at a time $t_0 \in I$ by

$$u^{[t_0]}(s, x) := N(t_0)^{-\frac{d-2}{2}} u\big(t_0 + N(t_0)^{-2}s, x(t_0) + N(t_0)^{-1}x\big). \tag{8.3}$$

Note that $u^{[t_0]}$ is a normalised solution which is almost periodic modulo symmetries with lifespan $I^{[t_0]} := \{s \in \mathbb{R} : t_0 + N(t_0)^{-2}s \in I\}$. The parameters of $u^{[t_0]}$ satisfy

$$N^{[t_0]}(s) := \frac{N\big(t_0 + sN(t_0)^{-2}\big)}{N(t_0)} \quad \text{and} \quad x^{[t_0]}(s) := N(t_0)\big[x\big(t_0 + sN(t_0)^{-2}\big) - x(t_0)\big]$$

and $u^{[t_0]}$ has the same compactness modulus function as u. Furthermore, if u is a maximal-lifespan solution then so is $u^{[t_0]}$.

Lemma 8.3 (Local constancy of $N(t)$ and $x(t)$, [20, 22])**.** *Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a non-zero almost periodic modulo symmetries solution to* (6.1) *with parameters $N(t)$ and $x(t)$. Then there exists a small number δ, depending on u, such that for every $t_0 \in I$ we have*

$$\big[t_0 - \delta N(t_0)^{-2}, t_0 + \delta N(t_0)^{-2}\big] \subset I \tag{8.4}$$

and

$$N(t) \sim_u N(t_0) \quad \textit{and} \quad |x(t) - x(t_0)| \lesssim_u N(t_0)^{-1} \tag{8.5}$$

whenever $|t - t_0| \leq \delta N(t_0)^{-2}$.

Proof. We first prove (8.4). Arguing by contradiction, we assume (8.4) fails. Thus, there exist sequences $t_n \in I$ and $\delta_n \to 0$ such that $t_n + \delta_n N(t_n)^{-2} \notin I$ for all n. Then $u^{[t_n]}$ given by (8.3) are normalised solutions whose lifespans $I^{[t_n]}$ contain 0 but not δ_n. Invoking almost periodicity and passing to a subsequence, we conclude that $u^{[t_n]}(0)$ converge to some $v_0 \in \dot{H}^1_x$. Let $v : J \times \mathbb{R}^d \to \mathbb{C}$ be the maximal-lifespan solution with data $v(0) = v_0$. By the local well-posedness theory, J is an open interval and so contains δ_n for all sufficiently large n. By the stability result Theorem 5.3, for n sufficiently large we must have that $\delta_n \in I^{[t_n]}$. This contradicts the hypothesis and so gives (8.4).

We now turn to (8.5). Again, we argue by contradiction, taking δ even smaller if necessary. Suppose one of the two claims in (8.5) failed no matter how small one chose δ. Then one can find sequences $t_n, t_n' \in I$ so that $s_n := (t_n' - t_n)N(t_n)^2 \to 0$ but $N(t_n')/N(t_n)$ converge to either zero or infinity (if the first claim failed) or $|x(t_n') - x(t_n)|N(t_n) \to \infty$ (if the second claim failed). Therefore, $N^{[t_n]}(s_n)$ converge to either zero or infinity or $x^{[t_n]}(s_n) \to \infty$. By almost periodicity, this implies that $u^{[t_n]}(s_n)$ must converge weakly to zero.

On the other hand, using almost periodicity and passing to a subsequence we find that $u^{[t_n]}(0)$ converge to some $v_0 \in \dot{H}^1_x$. As $s_n \to 0$, we conclude that $u^{[t_n]}(s_n)$ converge to v_0 in $\dot{H}^1_x$. Thus $v_0 = 0$ and $E(u) = E(u^{[t_n]}) \to E(v_0) = 0$. This means $u \equiv 0$, a contradiction. This completes the proof of (8.5). $\square$

An immediate consequence of Lemma 8.3 is the following corollary, which describes the behaviour of the frequency scale function.

Corollary 8.4 ($N(t)$ at blowup, [20, 22]). *Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a non-zero maximal-lifespan solution to* (6.1) *that is almost periodic modulo symmetries with frequency scale function $N : I \to \mathbb{R}^+$. If T is any finite endpoint of the lifespan I, then $N(t) \gtrsim_u |T-t|^{-1/2}$; in particular, $\lim_{t\to T} N(t) = \infty$. If I is infinite or semi-infinite, then for any $t_0 \in I$ we have $N(t) \gtrsim_u \min\{N(t_0), |t-t_0|^{-1/2}\}$.*

Proof. Exercise! □

Our next result shows how energy-critical norms of an almost periodic solution can be computed in terms of its frequency scale function; see [20] for the mass-critical analogue.

Lemma 8.5 (Strichartz norms via $N(t)$, [22]). *Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a non-zero almost periodic modulo symmetries solution to* (6.1) *with frequency scale function $N : I \to \mathbb{R}^+$. Then*

$$\int_I N(t)^2\,dt \lesssim_u \int_I \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+2)}{d-2}}\,dx\,dt \lesssim_u 1 + \int_I N(t)^2\,dt.$$

Proof. We first prove

$$\int_I \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+2)}{d-2}}\,dx\,dt \lesssim_u 1 + \int_I N(t)^2\,dt. \tag{8.6}$$

Let $0 < \eta < 1$ be a small parameter to be chosen shortly and partition I into subintervals I_j so that

$$\int_{I_j} N(t)^2\,dt \le \eta; \tag{8.7}$$

this requires at most $\eta^{-1} \times \mathrm{RHS}(8.6)$ many intervals.

For each j, we may choose $t_j \in I_j$ so that

$$N(t_j)^2 |I_j| \le 2\eta. \tag{8.8}$$

By Sobolev embedding, Strichartz, Hölder, and Bernstein, we obtain

$$\begin{aligned} \|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} &\lesssim \|\nabla u\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}} \\ &\lesssim \|e^{i(t-t_j)\Delta}\nabla u(t_j)\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}} + \|\nabla u\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}}^{\frac{d+2}{d-2}} \\ &\lesssim \|u_{\ge N_0}(t_j)\|_{\dot H_x^1} + |I_j|^{\frac{d-2}{2(d+2)}} N_0^{\frac{d-2}{d+2}} \|u(t_j)\|_{\dot H_x^1} + \|\nabla u\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}}^{\frac{d+2}{d-2}}, \end{aligned}$$

where all spacetime norms are over $I_j \times \mathbb{R}^d$. Choosing N_0 as a large multiple of $N(t_j)$ and using almost periodicity modulo symmetries, we can make the first term as small as we wish. Subsequently, choosing η sufficiently small depending on $E(u)$ and invoking (8.8), we may also render the second term arbitrarily small. Thus, by the usual bootstrap argument we obtain

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I_j\times\mathbb{R}^d)} \lesssim \|\nabla u\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}(I_j\times\mathbb{R}^d)} \le 1.$$

Using the bound on the number of intervals I_j, this leads to (8.6).

Next we prove

$$\int_I \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+2)}{d-2}}\,dx\,dt \gtrsim_u \int_I N(t)^2\,dt. \tag{8.9}$$

Using almost periodicity and Sobolev embedding, we can guarantee that

$$\int_{|x-x(t)|\leq C(u)N(t)^{-1}} |u(t,x)|^{\frac{2d}{d-2}}\,dx \gtrsim_u 1 \tag{8.10}$$

uniformly for $t\in I$. On the other hand, by Hölder,

$$\int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+2)}{d-2}}\,dx \gtrsim_u \Bigl(\int_{|x-x(t)|\leq C(u)N(t)^{-1}} |u(t,x)|^{\frac{2d}{d-2}}\,dx\Bigr)^{\frac{d+2}{d}} N(t)^2.$$

Using (8.10) and integrating over I we obtain (8.9). □

Corollary 8.6. *Let $u: I\times\mathbb{R}^d\to\mathbb{C}$ be a non-zero almost periodic modulo symmetries solution to* (6.1) *with frequency scale function $N: I\to\mathbb{R}^+$. Then*

$$\|\nabla u\|^2_{L_t^2 L_x^{\frac{2d}{d-2}}(I\times\mathbb{R}^d)} \lesssim_u 1 + \int_I N(t)^2\,dt.$$

Proof. Exercise! □

The next proposition tells us that for a minimal blowup solution $u: I\times\mathbb{R}^d\to\mathbb{C}$, the free evolution coming for the endpoints of the maximal-lifespan I converges weakly to zero in $\dot H^1_x$. Intuitively, we expect this to be the case since the free evolution is nothing but radiation and radiation does not directly contribute to blowup. However, a minimal blowup solution needs all its norm in order to blow up and so cannot waste any norm on a radiation term.

Proposition 8.7 (Reduced Duhamel formulas, [22, 36])**.** *Let $u: I\times\mathbb{R}^d\to\mathbb{C}$ be a maximal-lifespan almost periodic modulo symmetries solution to* (6.1)*. Then $e^{-it\Delta}u(t)$ converges weakly to zero in $\dot H^1_x$ as $t\to\sup I$ or $t\to\inf I$. In particular, we have the 'reduced' Duhamel formulas*

$$\begin{aligned} u(t) &= i\lim_{T\to\sup I}\int_t^T e^{i(t-s)\Delta}|u(s)|^{\frac{4}{d-2}}u(s)\,ds \\ &= -i\lim_{T\to\inf I}\int_T^t e^{i(t-s)\Delta}|u(s)|^{\frac{4}{d-2}}u(s)\,ds, \end{aligned} \tag{8.11}$$

where the limits are to be understood in the weak $\dot H^1_x$ topology.

Proof. We prove the claim as $t\to\sup I$; the proof in the reverse time direction is similar.

Assume first that $\sup I<\infty$. Then by Corollary 8.4,

$$\lim_{t\to\sup I} N(t)=\infty.$$

By almost periodicity, this implies that $u(t)$ converges weakly to zero as $t\to\sup I$. As $\sup I<\infty$ and the map $t\mapsto e^{it\Delta}$ is continuous in the strong operator topology on $\dot H^1_x$, we see that $e^{-it\Delta}u(t)$ converges weakly to zero, as desired.

Now suppose that $\sup I=\infty$. We need to prove that

$$\lim_{t\to\infty}\langle u(t), e^{it\Delta}\phi\rangle_{\dot H^1_x}=0$$

for all test functions $\phi\in C_c^\infty(\mathbb{R}^d)$. Let $\eta>0$ be a small parameter. By Cauchy–Schwarz and almost periodicity,

$$\Bigl|\langle u(t), e^{it\Delta}\phi\rangle_{\dot H^1_x}\Bigr|^2 \lesssim \Bigl|\int_{|x-x(t)|\leq C(\eta)/N(t)} \nabla u(t,x)\overline{e^{it\Delta}\nabla\phi(x)}\,dx\Bigr|^2$$

$$+\left|\int_{|x-x(t)|\geq C(\eta)/N(t)} \nabla u(t,x)\overline{e^{it\Delta}\nabla\phi(x)}\,dx\right|^2$$
$$\lesssim \|u(t)\|_{\dot H^1_x}^2 \int_{|x-x(t)|\leq C(\eta)/N(t)} |e^{it\Delta}\nabla\phi(x)|^2\,dx + \eta\|\phi\|_{\dot H^1_x}^2.$$

Therefore, to obtain the claim we merely need to show that

$$\int_{|x-x(t)|\leq C(\eta)/N(t)} |e^{it\Delta}\nabla\phi(x)|^2\,dx \to 0 \quad \text{as} \quad \eta\to 0.$$

This follows from Lemma 8.8 below, Corollary 8.4, and a change of variables. □

Lemma 8.8 (Fraunhofer formula)**.** *For $\psi\in L^2(\mathbb{R}^d)$ and $t\to\pm\infty$,*

$$\big\|[e^{it\Delta}\psi](x) - (2it)^{-\frac d2} e^{i|x|^2/4t}\hat\psi\big(\tfrac{x}{2t}\big)\big\|_{L^2_x} \to 0. \tag{8.12}$$

Proof. This asymptotic is most easily understood in terms of stationary phase. However, our proof will be based on the exact formula for the Schrödinger propagator, which we derived in Section 2. We have the identity

$$\begin{aligned}\text{LHS}(8.12) &= \Big\|(4\pi it)^{-\frac d2}\int_{\mathbb{R}^d} e^{i|x-y|^2/4t}[1-e^{-i|y|^2/4t}]\psi(y)\,dy\Big\|_{L^2_x}\\ &= \big\|e^{it\Delta}\big[(1-e^{-i|\cdot|^2/4t})\psi\big]\big\|_{L^2_x}\\ &= \big\|(1-e^{-i|\cdot|^2/4t})\psi\big\|_{L^2_x}.\end{aligned}$$

The result now follows from the dominated convergence theorem. □

So far we have proved that if Theorem 6.1 fails, then there exists a minimal witness to its failure. This is a maximal-lifespan almost periodic solution $u: I\times\mathbb{R}^d\to\mathbb{C}$ which blows up in both time directions; see Theorem 8.1. Moreover, we have recorded some basic properties satisfied by the modulation parameters $N(t)$ and $x(t)$. Thus, to prove Theorem 6.1 we have to rule out the existence of these minimal counterexamples. In order to achieve this, we need more quantitative information regarding $N(t)$ and $x(t)$. The first modest step in this direction is the following theorem, which asserts that we may assume $N(t)$ is bounded from below; the price we pay for this information is that we can no longer guarantee that u blows up in *both* time directions.

For an argument that is upside down relative to the one we present below, see Theorem 3.3 in [35]. This reference treats the mass-critical NLS and restricts attention to almost periodic solutions with $N(t)\leq 1$.

Theorem 8.9. *Suppose Theorem 6.1 fails to be true. Then there exists an almost periodic modulo symmetries solution $u: I\times\mathbb{R}^d\to\mathbb{C}$ such that $S_I(u)=\infty$ and*

$$N(t)\geq 1 \quad \textit{for all} \quad t\in I. \tag{8.13}$$

Proof. By Theorem 8.1, there exists a maximal-lifespan solution $v: J\times\mathbb{R}^d\to\mathbb{C}$ to the defocusing energy-critical NLS which is almost periodic modulo symmetries and which blows up in both time directions in the sense that $S_{\geq 0}(v)=S_{\leq 0}(v)=\infty$. Let $N_v(t)$ denote the frequency scale function associated to v. We will obtain the desired u satisfying (8.13) from v, by rescaling appropriately.

Write J as a nested union of compact intervals $J_1 \subset J_2 \subset \ldots \subset J$. On each compact interval J_n, we have $v \in C_t \dot H^1_x(J_n \times \mathbb{R}^d)$, which easily implies that $N_v(t)$ is bounded above and below on J_n. Thus, we may find $t_n \in J_n$ such that

$$N_v(t_n) \le 2N_v(t) \quad \text{for all} \quad t \in J_n. \tag{8.14}$$

Now consider the normalizations $v^{[t_n]} : I_n \times \mathbb{R}^d \to \mathbb{C}$ with $I_n := \{t \in \mathbb{R} : t_n + N_v(t_n)^{-2} t \in J_n\}$. Using almost periodicity and passing to a subsequence, we get that $v^{[t_n]}(0)$ converge in $\dot H^1_x$ to some u_0. From the conservation of energy, we see that u_0 is not identically zero. Let $u : (-T_-, T_+) \times \mathbb{R}^d \to \mathbb{C}$ be the maximal-lifespan solution with data $u(0) = u_0$.

Now let $v_n : \tilde I_n \times \mathbb{R}^d \to \mathbb{C}$ be the maximal-lifespan solution which agrees with $v^{[t_n]}$ on I_n. If K is any compact subinterval of $(-T_-, T_+)$ containing 0, then $S_K(u) < \infty$. From the stability result Theorem 5.3, for sufficiently large n we must have $K \subseteq \tilde I_n$ and $S_K(v_n) < \infty$ uniformly in n. As $S_{J_n}(v) = S_{I_n}(v_n) \to \infty$ as $n \to \infty$, we must have $I_n \not\subseteq K$ for n large. Passing to subsequence if necessary, this leaves only two possibilities:

- For every $0 < t < T_+$, $[0, t] \subseteq I_n$ for all sufficiently large n.
- For every $-T_- < t < 0$, $[t, 0] \subseteq I_n$ for all sufficiently large n.

By time reversal symmetry, it suffices to consider the former possibility. Let $I := [0, T_+)$. We will prove that $u : I \times \mathbb{R}^d \to \mathbb{C}$ satisfies the conclusions of Theorem 8.9.

We first note that $u : I \times \mathbb{R}^d \to \mathbb{C}$ is almost periodic modulo symmetries. Indeed, for any $0 \le t < T_+$, $u(t)$ can be approximated to arbitrary accuracy in $\dot H^1_x$ by $v^{[t_n]}(t)$, which is a rescaled version of a function in the orbit $\{v(t) : t \in J\}$. As the orbit of v is precompact in $\dot H^1_x$ modulo symmetries, then so is $\{u(t) : 0 \le t < T_+\}$.

Next we prove that $S_I(u) = \infty$. Otherwise we would have $T_+ = \infty$ and $[0, \infty) \subseteq I_n$ for n large. Moreover, by the stability theory, for n large we get $S_{\ge 0}(v^{[t_n]}) = S_{\ge t_n}(v) < \infty$, which contradicts the fact that v blows up forward in time.

Finally, we prove (8.13). Let $\eta > 0$ to be chosen later. Fix $t \in I$. By the stability result, for n large we have $t \in I_n$ and

$$\|v^{[t_n]}(t) - u(t)\|_{\dot H^1_x} \to 0 \quad \text{as} \quad n \to \infty.$$

Combining this with (8.14) and almost periodicity, we find that there exists $c(\eta) > 0$ such that

$$\begin{aligned}
\eta &\ge \int_{|\xi| \le c(\eta) N_v(t)} |\xi \hat v(t, \xi)|^2 \, d\xi = \int_{|\xi| \le c(\eta) \frac{N_v(t_n + N_v(t_n)^{-2} t)}{N_v(t_n)}} |\xi \widehat{v^{[t_n]}}(t, \xi)|^2 \, d\xi \\
&\ge \int_{|\xi| \le \frac12 c(\eta)} |\xi \widehat{v^{[t_n]}}(t, \xi)|^2 \, d\xi \to \int_{|\xi| \le \frac12 c(\eta)} |\xi \hat u(t, \xi)|^2 \, d\xi.
\end{aligned}$$

Combining this with the definition of almost periodicity, we derive (8.13). This completes the proof of the theorem. □

Putting together the results of this section we can restrict attention to the following very specific enemy to Theorem 6.1:

Theorem 8.10. *Suppose Theorem 6.1 fails to be true. Then there exists an almost periodic solution $u : [0, T_{max}) \times \mathbb{R}^d \to \mathbb{C}$ such that*

$$S_{[0, T_{max})}(u) = \int_0^{T_{max}} \int_{\mathbb{R}^d} |u(t, x)|^{\frac{2(d+2)}{d-2}} \, dx \, dt = +\infty.$$

Moreover, we may write $[0, T_{max}) = \cup_k J_k$ *with* J_k *being intervals of local constancy and*

$$N(t) \equiv N_k \geq 1 \quad \textit{for each} \quad t \in J_k.$$

In the following two sections we will see how to preclude the existence of the almost periodic solution described in Theorem 8.10 for the defocusing energy-critical NLS in four spatial dimensions:

$$i\partial_t u = -\Delta u + |u|^2 u \quad \text{with} \quad u(0) = u_0 \in \dot{H}^1_x(\mathbb{R}^4). \tag{8.15}$$

Some of the arguments that follow work also in higher dimensions, as well as for the focusing equation; however, in these notes we are not aiming for the greatest generality, but rather we try to demonstrate how these techniques can be used to settle Theorem 6.1 in the particular case $d = 4$.

Before we launch into the involved argument that will preclude the existence of the enemy described in Theorem 8.10, let us first pause and collect the rewards of this section. In particular, we will see that our enemy must be global forward in time; strictly speaking this step is not necessary for the argument that follows, but it is always good to realize how far we have come and how much further there is to go.

Theorem 8.11. *Let* $u : [0, T_{max}) \times \mathbb{R}^4 \to \mathbb{C}$ *be an almost periodic solution to* (8.15) *with* $S_{[0,T_{max})}(u) = \infty$. *Then* $T_{max} = \infty$.

Proof. We argue by contradiction. Assume that $T_{max} < \infty$. Using Proposition 8.7, the Strichartz inequality, Hölder's inequality, and the conservation of energy, we estimate

$$\begin{aligned}\|u_{\geq N}(t)\|_{L^2_x} \lesssim \|P_{\geq N}(|u|^2 u)\|_{L^2_t L^{4/3}_x([t,T_{max})\times\mathbb{R}^4)} &\lesssim (T_{max} - t)^{1/2} \|u\|^3_{L^\infty_t L^4_x([t,T_{max})\times\mathbb{R}^4)} \\ &\lesssim_u (T_{max} - t)^{1/2},\end{aligned}$$

uniformly in $N \in 2^{\mathbb{Z}}$. Letting $N \to 0$ we deduce that u has finite mass; letting $t \to T_{max}$ and invoking the conservation of mass, we deduce that

$$M(u(t)) = \int_{\mathbb{R}^4} |u(t,x)|^2 \, dx = 0 \quad \text{for all} \quad t \in [0, T_{max}).$$

In particular, $u \equiv 0$, which contradicts the fact that $S_{[0,T_{max})}(u) = \infty$.

This completes the proof of the theorem. □

9. Long-time Strichartz estimates and applications

In this section, we prove a long-time Strichartz inequality for solutions to (8.15) as described in Theorem 8.10. This will then be used to rule out rapid frequency cascade solutions, namely, solutions which also satisfy

$$\int_0^{T_{\max}} N(t)^{-1} \, dt < \infty.$$

9.1. **A long-time Strichartz inequality.**

Long-time Strichartz inequalities originate in the work of Dodson [15] on the mass-critical NLS. The main result of this section is a long-time Strichartz estimate for solutions to (8.15). This was proved in [40]; we review the proof below.

Theorem 9.1 (Long-time Strichartz estimates)**.** *Let $u : [0, T_{max}) \times \mathbb{R}^4 \to \mathbb{C}$ be an almost periodic solution to* (8.15) *with $N(t) \equiv N_k \geq 1$ on each characteristic interval $J_k \subset [0, T_{max})$. Then, on any compact time interval $I \subset [0, T_{max})$, which is a union of contiguous intervals J_k, and for any frequency $M > 0$,*

$$\|\nabla u_{\leq M}\|_{L_t^2 L_x^4(I\times\mathbb{R}^4)} \lesssim_u 1 + M^{3/2} K^{1/2}, \tag{9.1}$$

where $K := \int_I N(t)^{-1}\, dt$. Moreover, for any $\eta > 0$ there exists $M_0 = M_0(\eta) > 0$ such that for all $M \leq M_0$,

$$\|\nabla u_{\leq M}\|_{L_t^2 L_x^4(I\times\mathbb{R}^4)} \lesssim_u \eta\big(1 + M^{3/2} K^{1/2}\big). \tag{9.2}$$

Importantly, the constant M_0 and the implicit constants in (9.1) *and* (9.2) *are independent of the interval I.*

Proof. Fix a compact time interval $I \subset [0, T_{max})$, which is a union of contiguous intervals J_k. Throughout the proof all spacetime norms will be on $I \times \mathbb{R}^4$, unless we specify otherwise. Let $\eta_0 > 0$ be a small parameter to be chosen later. By almost periodicity, there exists $c_0 = c_0(\eta_0)$ such that

$$\|\nabla u_{\leq c_0 N(t)}\|_{L_t^\infty L_x^2} \leq \eta_0. \tag{9.3}$$

For $M > 0$ we define

$$A(M) := \|\nabla u_{\leq M}\|_{L_t^2 L_x^4(I\times\mathbb{R}^4)}.$$

Note that Corollary 8.5 implies

$$A(M) \lesssim_u 1 + M^{3/2} K^{1/2} \quad \text{whenever} \quad M \geq \left(\frac{\int_I N(t)^2\, dt}{\int_I N(t)^{-1}\, dt}\right)^{1/3}, \tag{9.4}$$

and, in particular, whenever $M \geq N_{max} := \sup_{t\in I} N(t)$. We will obtain the result for arbitrary frequencies $M > 0$ by induction. Our first step is to obtain a recurrence relation for $A(M)$. We start with an application of the Strichartz inequality:

$$A(M) \lesssim \inf_{t\in I} \|\nabla u_{\leq M}(t)\|_{L_x^2} + \big\|\nabla P_{\leq M} F(u)\big\|_{L_t^2 L_x^{4/3}}. \tag{9.5}$$

To continue, we decompose $u = u_{\leq M/\eta_0} + u_{>M/\eta_0}$ and then further decompose $u(t) = u_{\leq c_0 N(t)}(t) + u_{>c_0 N(t)}(t)$. Thus we may write

$$F(u) = \emptyset\big(u_{>M/\eta_0} u^2\big) + \emptyset\big(\big(P_{\leq c_0 N(t)} u_{\leq M/\eta_0}\big)^3\big) + \emptyset\big(u^2_{\leq M/\eta_0} u_{>c_0 N(t)}\big), \tag{9.6}$$

where we use the notation $\emptyset(X)$ to denote a quantity that resembles X, that is, a finite linear combination of terms that look like those in X, but possibly with some factors replaced by their complex conjugates and/or restricted to various frequencies. Next, we will estimate the contributions of each of these terms to (9.5).

To estimate the contribution of the first term on the right-hand side of (9.6), we use the Bernstein inequality followed by Lemma A.9, Lemma A.6, Hölder, and Sobolev embedding:

$$\begin{aligned}
\big\|\nabla P_{\leq M} \emptyset\big(u_{>M/\eta_0} u^2\big)\big\|_{L_t^2 L_x^{4/3}} &\lesssim M^{5/3} \big\||\nabla|^{-2/3} \emptyset\big(u_{>M/\eta_0} u^2\big)\big\|_{L_t^2 L_x^{4/3}} \\
&\lesssim M^{5/3} \big\||\nabla|^{-2/3} u_{>M/\eta_0}\big\|_{L_t^2 L_x^4} \big\||\nabla|^{2/3} \emptyset\big(u^2\big)\big\|_{L_t^\infty L_x^{3/2}} \\
&\lesssim M^{5/3} \big\||\nabla|^{-2/3} u_{>M/\eta_0}\big\|_{L_t^2 L_x^4} \big\||\nabla|^{2/3} u\big\|_{L_t^\infty L_x^{12/5}} \|u\|_{L_t^\infty L_x^4} \\
&\lesssim M^{5/3} \big\||\nabla|^{-2/3} u_{>M/\eta_0}\big\|_{L_t^2 L_x^4} \|u\|^2_{L_t^\infty \dot{H}_x^1}
\end{aligned}$$

$$\lesssim_u \sum_{L>M/\eta_0} \Big(\frac{M}{L}\Big)^{5/3} A(L). \tag{9.7}$$

We turn now to the contribution of the second term on the right-hand side of (9.6). Employing Hölder and (9.3), we obtain

$$\begin{aligned} \big\|\nabla P_{\leq M} \emptyset\big(\big(P_{\leq c_0 N(t)} u_{\leq M/\eta_0}\big)^3\big)\big\|_{L_t^2 L_x^{4/3}} &\lesssim \|\nabla u_{\leq M/\eta_0}\|_{L_t^2 L_x^4} \|u_{\leq c_0 N(t)}\|_{L_t^\infty L_x^4}^2 \\ &\lesssim_u \eta_0^2 A\big(M/\eta_0\big). \end{aligned} \tag{9.8}$$

Finally, we consider the contribution of the third term on the right-hand side of (9.6). By Bernstein and then Hölder,

$$\begin{aligned} \big\|\nabla P_{\leq M} \emptyset\big(u_{\leq M/\eta_0}^2 u_{>c_0 N(t)}\big)\big\|_{L_t^2 L_x^{4/3}} &\lesssim M \|u_{\leq M/\eta_0}\|_{L_t^\infty L_x^4} \|u_{\leq M/\eta_0} u_{>c_0 N(t)}\|_{L_{t,x}^2} \\ &\lesssim_u M \|u_{\leq M/\eta_0} u_{>c_0 N(t)}\|_{L_{t,x}^2}. \end{aligned}$$

To continue, we decompose the time interval I into intervals of local constancy J_k and apply the bilinear Strichartz estimate Corollary 2.10 on each J_k. Note that by Lemma 8.5, Corollary 8.6, and Hölder's inequality, on each J_k we have

$$\|\nabla u\|_{L_t^2 L_x^4(J_k\times\mathbb{R}^4)} + \|\nabla F(u)\|_{L_{t,x}^{3/2}(J_k\times\mathbb{R}^4)} \lesssim_u 1 \quad \text{and hence} \quad \|\nabla u\|_{S_0^*(J_k)} \lesssim_u 1.$$

Thus, using also Bernstein's inequality,

$$\begin{aligned} \big\|u_{\leq M/\eta_0} u_{>c_0 N(t)}\big\|_{L_{t,x}^2(J_k\times\mathbb{R}^4)} &\lesssim \frac{(M/\eta_0)^{1/2}}{(c_0 N_k)^{1/2}} \|\nabla u_{\leq M/\eta_0}\|_{S_0^*(J_k)} \|u_{>c_0 N_k}\|_{S_0^*(J_k)} \\ &\lesssim_u \frac{M^{1/2}}{\eta_0^{1/2} c_0^{3/2} N_k^{3/2}} \|\nabla u_{\leq M/\eta_0}\|_{S_0^*(J_k)}. \end{aligned}$$

The term $\|\nabla u_{\leq M/\eta_0}\|_{S_0^*(J_k)}$ will be essential in obtaining the small parameter η in claim (9.2) and this is why we choose to keep it in the display above rather than discarding it. Summing the estimates above over the intervals J_k and invoking again the local constancy property Lemma 8.3, we find

$$\begin{aligned} \big\|u_{\leq M/\eta_0} u_{>c_0 N(t)}\big\|_{L_{t,x}^2(I\times\mathbb{R}^4)} &\lesssim_u \frac{M^{1/2}}{\eta_0^{1/2} c_0^{3/2}} \Big(\sum_{J_k\subset I} \frac{1}{N_k^3}\Big)^{1/2} \sup_{J_k\subset I} \|\nabla u_{\leq M/\eta_0}\|_{S_0^*(J_k)} \\ &\lesssim_u \frac{M^{1/2} K^{1/2}}{\eta_0^{1/2} c_0^{3/2}} \sup_{J_k\subset I} \|\nabla u_{\leq M/\eta_0}\|_{S_0^*(J_k)}. \end{aligned}$$

Thus, the contribution of the third term on the right-hand side of (9.6) can be bounded as follows:

$$\big\|\nabla P_{\leq M} \emptyset\big(u_{\leq M/\eta_0}^2 u_{>c_0 N(t)}\big)\big\|_{L_t^2 L_x^{4/3}} \lesssim_u \frac{M^{3/2} K^{1/2}}{\eta_0^{1/2} c_0^{3/2}} \sup_{J_k\subset I} \|\nabla u_{\leq M/\eta_0}\|_{S_0^*(J_k)}. \tag{9.9}$$

Collecting (9.5) through (9.9), we obtain

$$\begin{aligned} A(M) \lesssim_u \inf_{t\in I} \|\nabla u_{\leq M}(t)\|_{L_x^2} &+ \frac{M^{3/2} K^{1/2}}{\eta_0^{1/2} c_0^{3/2}} \sup_{J_k\subset I} \|\nabla u_{\leq M/\eta_0}\|_{S_0^*(J_k)} \\ &+ \sum_{L\geq \frac{M}{\eta_0}} \Big(\frac{M}{L}\Big)^{5/3} A(L). \end{aligned} \tag{9.10}$$

The inductive step in the proof of claims (9.1) and (9.2) will rely on this recurrence relation.

Let us first address (9.1). Recall that by (9.4), the claim holds for $M \geq N_{max}$, that is,

$$A(M) \leq C(u)\big[1 + M^{3/2}K^{1/2}\big], \tag{9.11}$$

for some constant $C(u) > 0$ and all $M \geq N_{max}$. Now using the fact that (9.10) implies

$$A(M) \leq \tilde{C}(u)\Big\{1 + \frac{M^{3/2}K^{1/2}}{\eta_0^{1/2}c_0^{3/2}} + \sum_{L \geq \frac{M}{\eta_0}} \Big(\frac{M}{L}\Big)^{5/3} A(L)\Big\}, \tag{9.12}$$

we can inductively prove the claim by halving the frequency M at each step. For example, assuming that (9.11) holds for frequencies larger or equal to M, an application of (9.12) (with $\eta_0 \leq 1/2$) yields

$$\begin{aligned} A\big(M/2\big) &\leq \tilde{C}(u)\Big\{1 + \frac{(M/2)^{3/2}K^{1/2}}{\eta_0^{1/2}c_0^{3/2}} + C(u)\sum_{L \geq \frac{M}{2\eta_0}} \Big(\frac{M}{2L}\Big)^{5/3}\big[1 + L^{3/2}K^{1/2}\big]\Big\} \\ &\leq \tilde{C}(u)\Big\{1 + \frac{(M/2)^{3/2}K^{1/2}}{\eta_0^{1/2}c_0^{3/2}} + 2\eta_0^{5/3}C(u) + 2\eta_0^{1/6}C(u)(M/2)^{3/2}K^{1/2}\Big\}. \end{aligned}$$

Choosing $\eta_0 = \eta_0(u)$ small enough so that $\eta_0^{1/6}\tilde{C}(u) \leq 1/4$, we thus obtain

$$A\big(M/2\big) \leq \frac{1}{2}C(u)\Big\{1 + (M/2)^{3/2}K^{1/2}\Big\} + \tilde{C}(u)\Big\{1 + \frac{(M/2)^{3/2}K^{1/2}}{\eta_0^{1/2}c_0^{3/2}}\Big\}.$$

The claim now follows by setting $C(u) \geq 2\tilde{C}(u)\eta_0^{-1/2}c_0^{-3/2}$.

Next we turn to (9.2). To exhibit the small constant η, we will need the following

Lemma 9.2 (Vanishing of the small frequencies)**.** *Under the assumptions of Theorem 9.1, we have*

$$f(M) := \|\nabla u_{\leq M}\|_{L_t^\infty L_x^2([0,T_{max})\times\mathbb{R}^4)} + \sup_{J_k \subset [0,T_{max})} \|\nabla u_{\leq M}\|_{S_0^*(J_k)} \to 0 \quad \textit{as} \quad M \to 0.$$

Proof. As by hypothesis $\inf_{t\in[0,T_{max})} N(t) \geq 1$, almost periodicity yields

$$\lim_{M\to 0} \|\nabla u_{\leq M}\|_{L_t^\infty L_x^2([0,T_{max})\times\mathbb{R}^4)} = 0. \tag{9.13}$$

Now fix a characteristic interval $J_k \subset [0, T_{max})$ and recall that all Strichartz norms of u are bounded on J_k. In particular, we have

$$\|\nabla u\|_{L_t^2 L_x^4(J_k\times\mathbb{R}^4)} + \|u\|_{L_t^3 L_x^{12}(J_k\times\mathbb{R}^4)} + \|u\|_{L_{t,x}^6(J_k\times\mathbb{R}^4)} \lesssim_u 1.$$

Using this followed by the decomposition $u = u_{\leq M^{1/2}} + u_{>M^{1/2}}$, Hölder, and Bernstein, for any frequency $M > 0$ we estimate

$$\begin{aligned} \|\nabla u_{\leq M}\|_{S_0^*(J_k)} &= \|\nabla u_{\leq M}\|_{L_t^\infty L_x^2} + \|\nabla P_{\leq M}F(u)\|_{L_{t,x}^{3/2}} \\ &\lesssim \|\nabla u_{\leq M}\|_{L_t^\infty L_x^2} + \|\nabla P_{\leq M}F(u_{>M^{1/2}})\|_{L_{t,x}^{3/2}} + \|\nabla u_{>M^{1/2}}u_{\leq M^{1/2}}u\|_{L_{t,x}^{3/2}} \\ &\quad + \|\nabla u_{\leq M^{1/2}}u^2\|_{L_{t,x}^{3/2}} \\ &\lesssim \|\nabla u_{\leq M}\|_{L_t^\infty L_x^2} + M\|u_{>M^{1/2}}\|_{L_t^2 L_x^4}\|u_{>M^{1/2}}\|_{L_{t,x}^6}\|u_{>M^{1/2}}\|_{L_t^\infty L_x^4} \\ &\quad + \|\nabla u_{>M^{1/2}}\|_{L_t^2 L_x^4}\|u_{\leq M^{1/2}}\|_{L_t^\infty L_x^4}\|u\|_{L_{t,x}^6} + \|\nabla u_{\leq M^{1/2}}\|_{L_t^\infty L_x^2}\|u\|_{L_t^3 L_x^{12}}^2 \\ &\lesssim_u \|\nabla u_{\leq M}\|_{L_t^\infty L_x^2} + M^{1/2} + \|\nabla u_{\leq M^{1/2}}\|_{L_t^\infty L_x^2}. \end{aligned}$$

All spacetime norms in the estimates above are on $J_k \times \mathbb{R}^4$. As $J_k \subset [0, T_{max})$ was arbitrary, we find

$$\sup_{J_k \subset [0,T_{max})} \|\nabla u_{\leq M}\|_{S_0^*(J_k)} \lesssim_u M^{1/2} + \|\nabla u_{\leq M}\|_{L_t^\infty L_x^2([0,T_{max})\times\mathbb{R}^4)} + \|\nabla u_{\leq M^{1/2}}\|_{L_t^\infty L_x^2([0,T_{max})\times\mathbb{R}^4)}.$$

The claim now follows by combining this with (9.13). □

We are now ready to prove (9.2). Using (9.1) and Lemma 9.2, the estimate (9.10) implies

$$\begin{aligned} A(M) &\lesssim_u f(M) + \frac{M^{3/2}K^{1/2}}{\eta_0^{1/2} c_0^{3/2}} f(M) + \sum_{L \geq \frac{M}{\eta_0}} \Big(\frac{M}{L}\Big)^{5/3} A(L) \\ &\lesssim_u f(M) + \eta_0^{5/3} + \left\{ \frac{f(M)}{\eta_0^{1/2} c_0^{3/2}} + \eta_0^{1/6} \right\} M^{3/2} K^{1/2}. \end{aligned}$$

Thus, for any $\eta > 0$, choosing first $\eta_0 = \eta_0(\eta)$ such that $\eta_0^{1/6} \leq \eta$ and then $M_0 = M_0(\eta)$ such that $\frac{f(M_0)}{\eta_0^{1/2} c_0^{3/2}} \leq \eta$, we obtain

$$A(M) \lesssim_u \eta\big(1 + M^{3/2}K^{1/2}\big) \quad \text{for all} \quad M \leq M_0.$$

This completes the proof of Theorem 9.1. □

Next, we record a consequence of Theorem 9.1, which will be useful in the derivation of a frequency-localized interaction Morawetz inequality.

Corollary 9.3 (Low and high frequencies control)**.** *Let $u : [0, T_{max}) \times \mathbb{R}^4 \to \mathbb{C}$ be an almost periodic solution to (8.15) with $N(t) \equiv N_k \geq 1$ on each characteristic interval $J_k \subset [0, T_{max})$. Then, on any compact time interval $I \subset [0, T_{max})$, which is a union of contiguous intervals J_k, and for any frequency $M > 0$,*

(9.14)
$$\|u_{\geq M}\|_{L_t^q L_x^r(I\times\mathbb{R}^4)} \lesssim_u M^{-1}(1 + M^3K)^{\frac{1}{q}} \quad \textit{for all } \tfrac{1}{q} + \tfrac{2}{r} = 1 \textit{ with } 3 < q \leq \infty.$$

Moreover, for any $\eta > 0$ there exists $M_0 = M_0(\eta)$ such that for all $M \leq M_0$ we have

$$\|\nabla u_{\leq M}\|_{L_t^q L_x^r(I\times\mathbb{R}^4)} \lesssim_u \eta(1 + M^3K)^{\frac{1}{q}} \quad \textit{for all } \tfrac{1}{q} + \tfrac{2}{r} = 1 \textit{ with } 2 \leq q \leq \infty. \tag{9.15}$$

The constant M_0 and the implicit constants in (9.14) *and* (9.15) *are independent of the interval I.*

Proof. We first address (9.14). By (9.1) and Bernstein's inequality, for any $\varepsilon > 0$ and any frequency $M > 0$ we have

$$\begin{aligned} \big\||\nabla|^{-1/2-\varepsilon} u_{\geq M}\big\|_{L_t^2 L_x^4(I\times\mathbb{R}^4)} &\lesssim \sum_{L\geq M} L^{-3/2-\varepsilon} \|\nabla u_L\|_{L_t^2 L_x^4(I\times\mathbb{R}^4)} \\ &\lesssim_u \sum_{L\geq M} L^{-3/2-\varepsilon}(1 + L^{3/2}K^{1/2}) \\ &\lesssim_u M^{-3/2-\varepsilon}(1 + M^3K)^{1/2}. \end{aligned}$$

The claim now follows by interpolating with the energy bound:

$$\|u_{\geq M}\|_{L_t^q L_x^r(I\times\mathbb{R}^4)} \lesssim \big\||\nabla|^{-\frac{1}{2}-\frac{q-3}{2}} u_{\geq M}\big\|_{L_t^2 L_x^4(I\times\mathbb{R}^4)}^{2/q} \|\nabla u_{\geq M}\|_{L_t^\infty L_x^2(I\times\mathbb{R}^4)}^{1-2/q}$$

$$\lesssim_u M^{-1}(1+M^3K)^{1/q},$$

whenever $\frac{1}{q}+\frac{2}{r}=1$ and $3<q\leq\infty$.

We turn now to (9.15). As $\inf_{t\in I} N(t)\geq 1$, using almost periodicity, for any $\eta>0$ we can find $M_0(\eta)$ such that for all $M\leq M_0$,

$$\|\nabla u_{\leq M}\|_{L_t^\infty L_x^2(I\times\mathbb{R}^4)}\leq\eta.$$

The claim follows by interpolating with (9.2). $\square$

9.2. **The rapid frequency cascade scenario.** In this subsection, we preclude the existence of almost periodic solutions as in Theorem 8.10 for which $\int_0^{T_{max}} N(t)^{-1}\,dt<\infty$. We will show their existence is inconsistent with the conservation of mass.

Theorem 9.4 (No rapid frequency cascades)**.** *There are no almost periodic solutions $u:[0,T_{max})\times\mathbb{R}^4\to\mathbb{C}$ to (8.15) with $N(t)\equiv N_k\geq 1$ on each characteristic interval $J_k\subset[0,T_{max})$ such that $\|u\|_{L^6_{t,x}([0,T_{max})\times\mathbb{R}^4)}=+\infty$ and*

$$\int_0^{T_{max}} N(t)^{-1}\,dt<\infty. \tag{9.16}$$

Proof. We argue by contradiction. Let u be such a solution. Then by Corollary 8.4,

$$\lim_{t\to T_{max}} N(t)=\infty, \tag{9.17}$$

whether T_{max} is finite or infinite. Thus, by almost periodicity we have

$$\lim_{t\to T_{max}}\|\nabla u_{\leq M}(t)\|_{L_x^2}=0 \quad\text{for any}\quad M>0. \tag{9.18}$$

Now let I_n be a nested sequence of compact subintervals of $[0,T_{max})$ that are unions of contiguous characteristic intervals J_k. On each I_n we may now apply Theorem 9.1. Specifically, using (9.10) together with the hypothesis (9.16), we get

$$\begin{aligned}
A_n(M) &:= \|\nabla u_{\leq M}\|_{L_t^2 L_x^4(I_n\times\mathbb{R}^4)}\\
&\lesssim_u \inf_{t\in I_n}\|\nabla u_{\leq M}(t)\|_{L_x^2}+\frac{M^{3/2}}{\eta_0^{1/2}c_0^{3/2}}\Big[\int_0^{T_{max}} N(t)^{-1}\,dt\Big]^{1/2}+\sum_{L\geq\frac{M}{\eta_0}}\Big(\frac{M}{L}\Big)^{5/3}A_n(L)\\
&\lesssim_u \inf_{t\in I_n}\|\nabla u_{\leq M}(t)\|_{L_x^2}+\frac{M^{3/2}}{\eta_0^{1/2}c_0^{3/2}}+\sum_{L\geq\frac{M}{\eta_0}}\Big(\frac{M}{L}\Big)^{5/3}A_n(L)
\end{aligned}$$

for all frequencies $M>0$. Arguing as for (9.1), we find

$$\|\nabla u_{\leq M}\|_{L_t^2 L_x^4(I_n\times\mathbb{R}^4)}\lesssim_u \inf_{t\in I_n}\|\nabla u_{\leq M}(t)\|_{L_x^2}+M^{3/2}\quad\text{for all}\quad M>0.$$

Letting n tend to infinity and invoking (9.18), we obtain

$$\|\nabla u_{\leq M}\|_{L_t^2 L_x^4([0,T_{max})\times\mathbb{R}^4)}\lesssim_u M^{3/2}\quad\text{for all}\quad M>0. \tag{9.19}$$

Our next claim is that (9.19) implies

$$\|\nabla u_{\leq M}\|_{L_t^\infty L_x^2([0,T_{max})\times\mathbb{R}^4)}\lesssim_u M^{3/2}\quad\text{for all}\quad M>0. \tag{9.20}$$

Fix $M>0$. Using the Duhamel formula from Proposition 8.7 together with the Strichartz inequality, the decomposition $u=u_{\leq M}+u_{>M}$, Lemma A.9, Lemma A.6, (9.19), Bernstein, Hölder, and Sobolev embedding, we find

$$\|\nabla u_{\leq M}\|_{L_t^\infty L_x^2}\lesssim\|\nabla P_{\leq M}F(u)\|_{L_t^2 L_x^{4/3}}$$

$$
\begin{aligned}
&\lesssim \|\nabla P_{\le M} F(u_{\le M})\|_{L_t^2 L_x^{4/3}} + \|\nabla P_{\le M} \emptyset(u_{>M} u^2)\|_{L_t^2 L_x^{4/3}} \\
&\lesssim \|\nabla u_{\le M}\|_{L_t^2 L_x^4} \|u_{\le M}\|_{L_t^\infty L_x^4}^2 + M^{5/3} \big\||\nabla|^{-2/3} \emptyset(u_{>M} u^2)\big\|_{L_t^2 L_x^{4/3}} \\
&\lesssim_u M^{3/2} + M^{5/3} \big\||\nabla|^{-2/3} u_{>M}\big\|_{L_t^2 L_x^4} \big\||\nabla|^{2/3} u\big\|_{L_t^\infty L_x^{12/5}} \|u\|_{L_t^\infty L_x^4} \\
&\lesssim_u M^{3/2} + M^{5/3} \sum_{L>M} L^{-5/3} \|\nabla u_L\|_{L_t^2 L_x^4} \\
&\lesssim_u M^{3/2} + M^{5/3} \sum_{L>M} L^{-1/6} \\
&\lesssim_u M^{3/2}.
\end{aligned}
$$

All spacetime norms in the estimates above are on $[0, T_{max}) \times \mathbb{R}^4$.

With (9.20) in place, we are now ready to finish the proof of Theorem 9.4. First note that by Bernstein's inequality and (9.20), $u \in L_t^\infty \dot{H}_x^{-1/4}([0, T_{max}) \times \mathbb{R}^4)$; indeed,

$$
\begin{aligned}
\big\||\nabla|^{-1/4} u\|_{L_t^\infty L_x^2} &\lesssim \big\||\nabla|^{-1/4} u_{>1}\|_{L_t^\infty L_x^2} + \big\||\nabla|^{-1/4} u_{\le 1}\|_{L_t^\infty L_x^2} \\
&\lesssim_u \sum_{M>1} M^{-5/4} + \sum_{M \le 1} M^{1/4} \lesssim_u 1.
\end{aligned}
$$

Now fix $t \in [0, T_{max})$ and let $\eta > 0$ be a small constant. By almost periodicity, there exists $c(\eta) > 0$ such that

$$
\int_{|\xi| \le c(\eta) N(t)} |\xi|^2 |\hat{u}(t, \xi)|^2 \, d\xi \le \eta.
$$

Interpolating with $u \in L_t^\infty \dot{H}_x^{-1/4}$, we find

$$
\int_{|\xi| \le c(\eta) N(t)} |\hat{u}(t, \xi)|^2 \, d\xi \lesssim_u \eta^{1/5}. \tag{9.21}
$$

Meanwhile, by elementary considerations,

(9.22)

$$
\int_{|\xi| \ge c(\eta) N(t)} |\hat{u}(t, \xi)|^2 \, d\xi \le [c(\eta) N(t)]^{-2} \int_{\mathbb{R}^4} |\xi|^2 |\hat{u}(t, \xi)|^2 \, d\xi \lesssim_u [c(\eta) N(t)]^{-2}.
$$

Collecting (9.21) and (9.22) and using Plancherel's theorem, we obtain

$$
0 \le M(u(t)) := \int_{\mathbb{R}^4} |u(t, x)|^2 \, dx \lesssim_u \eta^{1/5} + c(\eta)^{-2} N(t)^{-2}
$$

for all $t \in [0, T_{max})$. Letting η tend to zero and invoking (9.17) and the conservation of mass, we conclude $M(u) = 0$ and hence u is identically zero. This contradicts $\|u\|_{L_{t,x}^6([0, T_{max}) \times \mathbb{R}^4)} = \infty$, thus settling Theorem 9.4. □

10. Frequency-localized interaction Morawetz inequalities and applications

Our goal in this section is to prove a frequency-localized interaction Morawetz inequality. This will then be used to preclude the existence of almost periodic solutions as in Theorem 8.10 for which $\int_0^{T_{max}} N(t)^{-1} \, dt = \infty$. These results appear in [40]; we review the proof below.

Before we delve into the gory details, let us pause to assess where we are. In view of Theorems 8.11 and 9.4, the only enemy we are left to face is an almost periodic solution $u : [0,\infty) \times \mathbb{R}^4 \to \mathbb{C}$ to (8.15) with $N(t) \equiv N_k \geq 1$ on each characteristic interval $J_k \subset [0,\infty)$ such that $\|u\|_{L^6_{t,x}([0,\infty)\times\mathbb{R}^4)} = +\infty$ and

$$\int_0^\infty N(t)^{-1}\,dt < \infty.$$

In order to rule out this quasi-soliton solution we need tools that express the defocusing nature of the equation. These are the various versions of the Morawetz inequality.

The Morawetz inequality originates in classical mechanics: in the presence of a repulsive potential, the quantity $\mathbf{p}(t)\cdot\frac{\mathbf{x}(t)}{|\mathbf{x}(t)|}$ is monotone. Here $\mathbf{p}$ denotes the momentum of the particle and $\mathbf{x}$ denotes its position. The natural quantum mechanical analogue of the quantity $\mathbf{p}(t)\cdot\frac{\mathbf{x}(t)}{|\mathbf{x}(t)|}$ is the Morawetz action

$$M(t) := 2\,\mathrm{Im}\int_{\mathbb{R}^4} \overline{u(t,x)}\nabla u(t,x)\cdot\frac{x}{|x|}\,dx,$$

where u is a solution to (8.15). A direct computation shows that

$$\partial_t M(t) \geq 2\int_{\mathbb{R}^4}\frac{|u(t,x)|^2}{|x|^3}\,dx + 3\int_{\mathbb{R}^4}\frac{|u(t,x)|^4}{|x|}\,dx.$$

Integrating with respect to time and using Cauchy–Schwarz we derive the Lin–Strauss Morawetz inequality, [25]:

$$\int_I\int_{\mathbb{R}^4}\frac{|u(t,x)|^4}{|x|}\,dx\,dt \lesssim \|u\|_{L^\infty_t L^2_x(I\times\mathbb{R}^4)}\|u\|_{L^\infty_t \dot H^1_x(I\times\mathbb{R}^4)}. \tag{10.1}$$

There are two obvious drawbacks when attempting to use this formula to preclude our final enemy. The first one is that it favours the origin: it basically says that if the solution is in $L^\infty_t H^1_x$, then it cannot spend a lot of time near the spatial origin. Secondly, in order to exploit inequality (10.1), we need the solution to lie in $L^\infty_t H^1_x$. However, even if we only cared about Schwartz solutions, when we apply the concentration compactness argument to exhibit a minimal counterexample to Theorem 6.1, we lose all information about the solution that is not left invariant by the symmetries of the equation; in particular, we are left with a solution that is merely in $L^\infty_t \dot H^1_x$.

Bourgain [5] showed us how to resolve the second issue above. His solution was to truncate in space; this is equivalent to throwing away the low frequencies of the solution. (Incidentally, truncating an $L^\infty_t \dot H^1_x$ solution to high frequencies places it in $L^\infty_t H^1_x$, although the truncation will no longer be a solution.) In this way, Bourgain obtained the following Morawetz inequality:

$$\int_I\int_{|x|\leq A|I|^{1/2}}\frac{|u(t,x)|^4}{|x|}\,dx\,dt \lesssim A|I|^{1/2}\|u\|^2_{L^\infty_t \dot H^1_x(I\times\mathbb{R}^4)}. \tag{10.2}$$

Compared with (10.1), it still favours the spatial origin, but at least now we can control the right-hand side.

Let us quickly see how to use (10.2) to complete the proof of Theorem 6.1 for *radial* initial data in dimension $d = 4$:

Step 1: We note that by rotation invariance and uniqueness of solutions to (8.15), solutions with radial initial data are radial for all time.

Step 2: Radial almost periodic solutions must concentrate near the spatial origin. Indeed, if $|x(t)| \gg N(t)^{-1}$, then by spherical symmetry there exist a very large number of disjoint balls on which $u(t)$ concentrates a nontrivial portion of its energy. This however contradicts the conservation of energy. Thus we must have $|x(t)| \lesssim N(t)^{-1}$. At this point we may set $x(t) \equiv 0$ by modifying the compactness modulus function accordingly.

Step 3: By Sobolev embedding and almost periodicity, we can find $C(u) > 0$ such that

$$\int_{|x| \leq C(u)/N(t)} |u(t,x)|^4 \, dx \gtrsim_u 1 \quad \text{uniformly for } t \in [0, \infty).$$

Step 4: Using (10.2) and Step 3 above, for any time interval $I \subset [0,\infty)$ which is a contiguous union of intervals of local constancy J_k we obtain

$$\begin{aligned}
|I|^{1/2} &\gtrsim_u \int_I \int_{|x| \leq C(u)|I|^{1/2}} \frac{|u(t,x)|^4}{|x|} \, dx \, dt \\
&\gtrsim_u \sum_{J_k \subset I} \int_{J_k} \int_{|x| \leq C(u)|J_k|^{1/2}} \frac{|u(t,x)|^4}{|x|} \, dx \, dt \\
&\gtrsim_u \sum_{J_k \subset I} \int_{J_k} \int_{|x| \leq C(u)/N(t)} N(t)|u(t,x)|^4 \, dx \, dt \\
&\gtrsim_u \sum_{J_k \subset I} \int_{J_k} N(t) \, dt \\
&\gtrsim_u \int_I N(t) \, dt.
\end{aligned}$$

Recalling that $\inf_{t \in [0,\infty)} N(t) \geq 1$, we derive a contradiction by taking the interval $I \subset [0,\infty)$ sufficiently large.

This completes the proof of Theorem 6.1 for radial initial data in dimension $d = 4$.

To handle nonradial initial data, Colliander–Keel–Staffilani–Takaoka–Tao [13] made use of an interaction Morawetz inequality, which they introduced in [12]. (Strictly speaking they treated the case $d = 3$. In what follows we consider the $d = 4$ analogue; see also [30].) Their idea was to center the Morawetz action not at the origin, but rather where the solution actually lives:

$$M_{interact}(t) := 2 \operatorname{Im} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \overline{u(t,x)} \nabla u(t,x) \cdot \frac{x-y}{|x-y|} |u(t,y)|^2 \, dx \, dy.$$

A computation gives

$$\partial_t M_{interact}(t) \gtrsim \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{|u(t,x)|^2 |u(t,y)|^2}{|x-y|^3} + \frac{|u(t,x)|^4 |u(t,y)|^2}{|x-y|} \, dx \, dy.$$

Thus, by the fundamental theorem of calculus and Cauchy–Schwarz,

$$\begin{aligned}
&\int_I \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{|u(t,x)|^2 |u(t,y)|^2}{|x-y|^3} + \frac{|u(t,x)|^4 |u(t,y)|^2}{|x-y|} \, dx \, dy \, dt \\
&\qquad \lesssim \|u\|^3_{L_t^\infty L_x^2(I \times \mathbb{R}^4)} \|u\|_{L_t^\infty \dot{H}_x^1(I \times \mathbb{R}^4)}.
\end{aligned} \tag{10.3}$$

This interaction Morawetz inequality has an obvious drawback, namely, in order to exploit it we need the solution to belong to $L_t^\infty H_x^1$. However, as noted before, our last enemy belongs merely to $L_t^\infty \dot{H}_x^1$. Therefore, in order to employ this new monotonicity formula, Colliander–Keel–Staffilani–Takaoka–Tao truncated the solution to frequencies greater than some frequency $N \in 2^{\mathbb{Z}}$, which is chosen small enough so that the truncation captures most of the norm of the solution. By almost periodicity, it is possible to chose N independent of time since our enemy satisfies $\inf_{t\in[0,\infty)} N(t) \geq 1$. Of course, since $u_{\geq N}$ no longer solves (8.15), there are additional errors introduced on the right-hand side of (10.3). Schematically, we obtain something of the form

$$\begin{aligned}\int_I \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{|u_{\geq N}(t,x)|^2 |u_{\geq N}(t,y)|^2}{|x-y|^3}\, dx\, dy\, dt \\ &\lesssim \|u_{\geq N}\|^3_{L_t^\infty L_x^2(I\times\mathbb{R}^4)} \|u_{\geq N}\|_{L_t^\infty \dot H_x^1(I\times\mathbb{R}^4)} + \text{errors} \\ &\lesssim_u N^{-3} + \text{errors}. \end{aligned} \tag{10.4}$$

If these errors were magically zero, then it would be a relatively easy task to use (10.4) to rule out our last enemy; see Theorem 10.3 below. However, these errors are not zero and controlling them is highly nontrivial.

Nowadays, there are two ways of handling the error terms on the right-hand side of (10.4). Colliander–Keel–Staffilani–Takaoka–Tao estimate these errors using solely the left-hand side in (10.4). The smallness needed to close the resulting bootstrap comes from the fact that $u_{\geq N}$ captures most of the norm of the solution and so $\|u_{\leq N}\|_{L_t^\infty \dot H_x^1} \ll 1$. A second approach inspired by Dodson's work on the mass-critical NLS is to first obtain additional a priori control in the form of the long-time Strichartz inequality we derived in Section 9; this is then used to control error terms in (10.4). It is this second approach that we will discuss here following [40]. This approach has also been adapted to the three dimensional problem originally treated by Colliander–Keel–Staffilani–Takaoka–Tao [13] in [21].

10.1. **A frequency-localized interaction Morawetz inequality.** In this subsection we derive a frequency-localized interaction Morawetz inequality, using the Dodson approach to control the error terms. We start by recalling the interaction Morawetz inequality in four spatial dimensions in slightly more generality; for details, see [30]. For a solution $\phi : I \times \mathbb{R}^4 \to \mathbb{C}$ to the equation $i\phi_t + \Delta\phi = \mathcal{N}$, we define the interaction Morawetz action

$$M_{interact}(t) := 2\,\mathrm{Im} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} |\phi(t,y)|^2 \frac{x-y}{|x-y|} \nabla\phi(t,x)\overline{\phi(t,x)}\, dx\, dy.$$

Standard computations show

$$\begin{aligned}\partial_t M_{interact}(t) \geq 3 \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{|\phi(t,x)|^2 |\phi(t,y)|^2}{|x-y|^3}\, dx\, dy \\ &+ 4\,\mathrm{Im} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \{\mathcal{N},\phi\}_m(t,y) \frac{x-y}{|x-y|} \nabla\phi(t,x)\overline{\phi(t,x)}\, dx\, dy \\ &+ 2 \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} |\phi(t,y)|^2 \frac{x-y}{|x-y|} \{\mathcal{N},\phi\}_p(t,x)\, dx\, dy, \end{aligned}$$

where the mass bracket is given by $\{\mathcal{N},\phi\}_m := \mathrm{Im}(\mathcal{N}\bar\phi)$ and the momentum bracket is given by $\{\mathcal{N},\phi\}_p := \mathrm{Re}(\mathcal{N}\nabla\bar\phi - \phi\nabla\bar{\mathcal{N}})$. Thus, integrating with respect to time, we obtain

Proposition 10.1 (Interaction Morawetz inequality)**.**

$$3 \int_I \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{|\phi(t,x)|^2 |\phi(t,y)|^2}{|x-y|^3}\, dx\, dy\, dt$$

$$
\begin{aligned}
&+ 2\int_I \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} |\phi(t,y)|^2 \frac{x-y}{|x-y|} \{\mathcal{N}, \phi\}_p(t,x)\, dx\, dy\, dt \\
&\leq 2\|\phi\|^3_{L_t^\infty L_x^2} \|\phi\|_{L_t^\infty \dot{H}_x^1} + 4\|\phi\|_{L_t^\infty L_x^2} \|\phi\|_{L_t^\infty \dot{H}_x^1} \|\{\mathcal{N}, \phi\}_m\|_{L^1_{t,x}},
\end{aligned}
$$

where all spacetime norms are over $I \times \mathbb{R}^4$.

We will apply Proposition 10.1 with $\phi = u_{\geq M}$ and $\mathcal{N} = P_{\geq M}(|u|^2 u)$ for M small enough that the Littlewood–Paley projection captures most of the solution. More precisely, we will prove

Proposition 10.2 (Frequency-localized interaction Morawetz estimate, [40])**.** *Let $u : [0, T_{max}) \times \mathbb{R}^4 \to \mathbb{C}$ be an almost periodic solution to* (8.15) *such that $N(t) \equiv N_k \geq 1$ on each characteristic interval $J_k \subset [0, T_{max})$. Then for any $\eta > 0$ there exists $M_0 = M_0(\eta)$ such that for $M \leq M_0$ and any compact time interval $I \subset [0, T_{max})$, which is a union of contiguous intervals J_k, we have*

$$
\int_I \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{|u_{\geq M}(t,x)|^2 |u_{\geq M}(t,y)|^2}{|x-y|^3}\, dx\, dy\, dt \lesssim_u \eta \Big[M^{-3} + \int_I N(t)^{-1}\, dt\Big].
$$

The implicit constant does not depend on the interval I.

Proof. Fix a compact interval $I \subset [0, T_{max})$, which is a union of contiguous intervals J_k, and let $K := \int_I N(t)^{-1}\, dt$. Throughout the proof, all spacetime norms will be on $I \times \mathbb{R}^4$.

Fix $\eta > 0$ and let $M_0 = M_0(\eta)$ be small enough that claim (9.15) of Corollary 9.3 holds; more precisely, for all $M \leq M_0$,

$$
\|\nabla u_{\leq M}\|_{L_t^q L_x^r} \lesssim_u \eta (1 + M^3 K)^{1/q} \quad \text{for all} \quad \tfrac{1}{q} + \tfrac{2}{r} = 1 \quad \text{with} \quad 2 \leq q \leq \infty. \tag{10.5}
$$

Choosing M_0 even smaller if necessary, we can also guarantee that

$$
\|u_{\geq M}\|_{L_t^\infty L_x^2} \lesssim_u \eta^6 M^{-1} \quad \text{for all} \quad M \leq M_0. \tag{10.6}
$$

Now fix $M \leq M_0$ and write $u_{lo} := u_{\leq M}$ and $u_{hi} := u_{>M}$. With this notation, (10.5) becomes

$$
\|\nabla u_{lo}\|_{L_t^q L_x^r} \lesssim_u \eta (1 + M^3 K)^{1/q} \quad \text{for all} \quad \tfrac{1}{q} + \tfrac{2}{r} = 1 \quad \text{with} \quad 2 \leq q \leq \infty. \tag{10.7}
$$

We will also need claim (9.14) of Corollary 9.3, which reads

$$
\|u_{hi}\|_{L_t^q L_x^r} \lesssim_u M^{-1} (1 + M^3 K)^{1/q} \quad \text{for all} \quad \tfrac{1}{q} + \tfrac{2}{r} = 1 \quad \text{with} \quad 3 < q \leq \infty. \tag{10.8}
$$

Note that by (10.6), the endpoint $q = \infty$ of the inequality above is strengthened to

$$
\|u_{hi}\|_{L_t^\infty L_x^2} \lesssim_u \eta^6 M^{-1}. \tag{10.9}
$$

To continue, we apply Proposition 10.1 with $\phi = u_{hi}$ and $\mathcal{N} = P_{hi} F(u)$ and use (10.9); we obtain

$$
\begin{aligned}
&\int_I \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{|u_{hi}(t,x)|^2 |u_{hi}(t,y)|^2}{|x-y|^3}\, dx\, dy\, dt \\
&\qquad + \int_I \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} |u_{hi}(t,y)|^2 \frac{x-y}{|x-y|} \{P_{hi} F(u), u_{hi}\}_p(t,x)\, dx\, dy\, dt \\
&\qquad\qquad \lesssim_u \eta^{18} M^{-3} + \eta^6 M^{-1} \|\{P_{hi} F(u), u_{hi}\}_m\|_{L^1_{t,x}(I \times \mathbb{R}^4)}.
\end{aligned} \tag{10.10}
$$

We first consider the contribution of the momentum bracket term. We write

$\{P_{hi} F(u), u_{hi}\}_p$

$$\begin{aligned}
&= \{F(u),u\}_p - \{F(u_{lo}),u_{lo}\}_p - \{F(u)-F(u_{lo}),u_{lo}\}_p - \{P_{lo}F(u),u_{hi}\}_p\\
&= -\tfrac{1}{2}\nabla[|u|^4 - |u_{lo}|^4] - \{F(u)-F(u_{lo}),u_{lo}\}_p - \{P_{lo}F(u),u_{hi}\}_p\\
&=: I + II + III.
\end{aligned}$$

After an integration by parts, the term I contributes to the left-hand side of (10.10) a multiple of

$$\int_I\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \frac{|u_{hi}(t,y)|^2|u_{hi}(t,x)|^4}{|x-y|}\,dx\,dy\,dt \\ + \sum_{j=1}^3 \int_I\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \frac{|u_{hi}(t,y)|^2|u_{hi}(t,x)|^j|u_{lo}(t,x)|^{4-j}}{|x-y|}\,dx\,dy\,dt.$$

In order to estimate the contribution of II to (10.10), we use $\{f,g\}_p = \nabla Ø(fg) + Ø(f\nabla g)$ to write

$$\{F(u)-F(u_{lo}),u_{lo}\}_p = \sum_{j=1}^3 \nabla Ø(u_{hi}^j u_{lo}^{4-j}) + \sum_{j=1}^3 Ø(u_{hi}^j u_{lo}^{3-j}\nabla u_{lo}).$$

Integrating by parts for the first term and bringing absolute values inside the integrals for the second term, we find that II contributes to the right-hand side of (10.10) a multiple of

$$\sum_{j=1}^3 \int_I\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \frac{|u_{hi}(t,y)|^2|u_{hi}(t,x)|^j|u_{lo}(t,x)|^{4-j}}{|x-y|}\,dx\,dy\,dt \\ + \sum_{j=1}^3 \int_I\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} |u_{hi}(t,y)|^2|u_{hi}(t,x)|^j|\nabla u_{lo}(t,x)||u_{lo}(t,x)|^{3-j}\,dx\,dy\,dt.$$

Finally, integrating by parts when the derivative (from the definition of the momentum bracket) falls on u_{hi}, we estimate the contribution of III to the right-hand side of (10.10) by a multiple of

$$\int_I\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \frac{|u_{hi}(t,y)|^2|u_{hi}(t,x)||P_{lo}F(u(t,x))|}{|x-y|}\,dx\,dy\,dt \\ + \int_I\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} |u_{hi}(t,y)|^2|u_{hi}(t,x)||\nabla P_{lo}F(u(t,x))|\,dx\,dy\,dt.$$

Consider now the mass bracket appearing in (10.10). Exploiting cancellation, we write

$$\begin{aligned}
&\{P_{hi}F(u),u_{hi}\}_m\\
&\quad= \{P_{hi}F(u) - F(u_{hi}),u_{hi}\}_m\\
&\quad= \{P_{hi}\big[F(u)-F(u_{hi})-F(u_{lo})\big],u_{hi}\}_m + \{P_{hi}F(u_{lo}),u_{hi}\}_m - \{P_{lo}F(u_{hi}),u_{hi}\}_m\\
&\quad= Ø(u_{hi}^3u_{lo}) + Ø(u_{hi}^2u_{lo}^2) + \{P_{hi}F(u_{lo}),u_{hi}\}_m - \{P_{lo}F(u_{hi}),u_{hi}\}_m.
\end{aligned}$$

Putting everything together and using (10.9), (10.10) becomes

(10.11)

$$\int_I\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \frac{|u_{hi}(t,x)|^2|u_{hi}(t,y)|^2}{|x-y|^3}\,dx\,dy\,dt + \int_I\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \frac{|u_{hi}(t,x)|^2|u_{hi}(t,y)|^4}{|x-y|}\,dx\,dy\,dt$$

$$\lesssim_u \eta^{18} M^{-3} \tag{10.12}$$

$$+ \eta^6 M^{-1}\big\{\|u_{hi}^3 u_{lo}\|_{L^1_{t,x}} + \|u_{hi}^2 u_{lo}^2\|_{L^1_{t,x}} + \|u_{hi} P_{hi} F(u_{lo})\|_{L^1_{t,x}} + \|u_{hi} P_{lo} F(u_{hi})\|_{L^1_{t,x}}\big\} \tag{10.13}$$

$$+ \eta^{12} M^{-2} \sum_{j=1}^{3} \|u_{hi}^j u_{lo}^{3-j} \nabla u_{lo}\|_{L^1_{t,x}} + \eta^{12} M^{-2} \|u_{hi} \nabla P_{lo} F(u)\|_{L^1_{t,x}} \tag{10.14}$$

$$+ \sum_{j=1}^{3} \int_I \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{|u_{hi}(t,y)|^2 |u_{hi}(t,x)|^j |u_{lo}(t,x)|^{4-j}}{|x-y|}\, dx\, dy\, dt \tag{10.15}$$

$$+ \int_I \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{|u_{hi}(t,y)|^2 |u_{hi}(t,x)| |P_{lo} F(u(t,x))|}{|x-y|}\, dx\, dy\, dt. \tag{10.16}$$

Thus, to complete the proof of Proposition 10.2 we have to show that the error terms (10.13) through (10.16) are acceptable; clearly, (10.12) is acceptable.

Consider now error term (10.13). Using (10.7), (10.8), and Sobolev embedding, we estimate

$$\|u_{hi}^3 u_{lo}\|_{L^1_{t,x}} \lesssim \|u_{hi}\|_{L^\infty_t L^4_x} \|u_{hi}\|^2_{L^{7/2}_t L^{14/5}_x} \|u_{lo}\|_{L^{7/3}_t L^{28}_x} \lesssim_u \eta M^{-2}(1 + M^3 K)$$

$$\|u_{hi}^2 u_{lo}^2\|_{L^1_{t,x}} \lesssim \|u_{hi}\|^2_{L^4_t L^{8/3}_x} \|u_{lo}\|^2_{L^4_t L^8_x} \lesssim_u \eta^2 M^{-2}(1 + M^3 K).$$

Using Bernstein's inequality as well, we estimate

$$\begin{aligned}\|u_{hi} P_{hi} F(u_{lo})\|_{L^1_{t,x}} &\lesssim \|u_{hi}\|_{L^4_t L^{8/3}_x} M^{-1} \|\nabla F(u_{lo})\|_{L^{4/3}_t L^{8/5}_x} \\ &\lesssim_u M^{-2}(1 + M^3 K)^{1/4} \|\nabla u_{lo}\|_{L^2_t L^4_x} \|u_{lo}\|^2_{L^8_t L^{16/3}_x} \\ &\lesssim_u \eta^3 M^{-2}(1 + M^3 K).\end{aligned}$$

Finally, by Hölder, Bernstein, Sobolev embedding, (10.7) and (10.8),

$$\begin{aligned}\|u_{hi} P_{lo} F(u_{hi})\|_{L^1_{t,x}} &\lesssim \|u_{hi}\|_{L^{10/3}_t L^{20/7}_x} M^{7/5} \|F(u_{hi})\|_{L^{10/7}_t L^1_x} \\ &\lesssim_u M^{2/5}(1 + M^3 K)^{3/10} \|u_{hi}\|^{7/3}_{L^{10/3}_t L^{20/7}_x} \|u_{hi}\|^{2/3}_{L^\infty_t L^{40/11}_x} \\ &\lesssim_u M^{2/5-7/3}(1 + M^3 K) \||\nabla|^{9/10} u_{hi}\|^{2/3}_{L^\infty_t L^2_x} \\ &\lesssim_u M^{-2}(1 + M^3 K).\end{aligned}$$

Collecting the estimates above we find

$$(10.13) \lesssim_u \eta^6 M^{-3}(1 + M^3 K) \lesssim_u \eta(M^{-3} + K),$$

and thus this error term is acceptable.

Consider next error term (10.14). By (10.7), (10.8), (10.9), Sobolev embedding, and Bernstein,

$$\|u_{hi} u_{lo}^2 \nabla u_{lo}\|_{L^1_{t,x}} \lesssim \|\nabla u_{lo}\|_{L^2_t L^4_x} \|u_{hi}\|_{L^\infty_t L^2_x} \|u_{lo}\|^2_{L^4_t L^8_x} \lesssim_u \eta^9 M^{-1}(1 + M^3 K)$$

$$\|u_{hi}^2 u_{lo} \nabla u_{lo}\|_{L^1_{t,x}} \lesssim \|\nabla u_{lo}\|_{L^2_t L^4_x} \|u_{hi}\|^2_{L^4_t L^{8/3}_x} \|u_{lo}\|_{L^\infty_{t,x}} \lesssim_u \eta^2 M^{-1}(1 + M^3 K)$$

$$\|u_{hi}^3 \nabla u_{lo}\|_{L^1_{t,x}} \lesssim \|\nabla u_{lo}\|_{L^{7/3}_t L^{28}_x} \|u_{hi}\|^2_{L^{7/2}_t L^{14/5}_x} \|u_{hi}\|_{L^\infty_t L^4_x} \lesssim_u \eta M^{-1}(1 + M^3 K).$$

To estimate the second term in (10.14), we write $F(u) = F(u_{lo}) + \O(u_{hi}u_{lo}^2 + u_{hi}^2 u_{lo} + u_{hi}^3)$. Arguing as above, we obtain

$$\begin{aligned}
\|u_{hi}\nabla P_{lo}F(u_{lo})\|_{L^1_{t,x}} &\lesssim \|u_{hi}\|_{L^\infty_t L^2_x}\|\nabla u_{lo}\|_{L^2_t L^4_x}\|u_{lo}\|^2_{L^4_t L^8_x} \lesssim_u \eta^9 M^{-1}(1+M^3K)\\
\|u_{hi}\nabla P_{lo}\O(u_{hi}u_{lo}^2)\|_{L^1_{t,x}} &\lesssim M\|u_{hi}\|^2_{L^4_t L^{8/3}_x}\|u_{lo}\|^2_{L^4_t L^8_x} \lesssim_u \eta^2 M^{-1}(1+M^3K)\\
\|u_{hi}\nabla P_{lo}\O(u_{hi}^2 u_{lo})\|_{L^1_{t,x}} &\lesssim M\|u_{hi}\|_{L^\infty_t L^4_x}\|u_{hi}\|^2_{L^{7/2}_t L^{14/5}_x}\|u_{lo}\|_{L^{7/3}_t L^{28}_x}\\
&\lesssim_u \eta M^{-1}(1+M^3K)\\
\|u_{hi}\nabla P_{lo}\O(u_{hi}^3)\|_{L^1_{t,x}} &\lesssim \|u_{hi}\|_{L^{10/3}_t L^{20/7}_x} M^{12/5}\|u_{hi}^3\|_{L^{10/7}_t L^1_x}\\
&\lesssim M^{12/5}\|u_{hi}\|^{10/3}_{L^{10/3}_t L^{20/7}_x}\|u_{hi}\|^{2/3}_{L^\infty_t L^{40/11}_x}\\
&\lesssim_u M^{-1}(1+M^3K).
\end{aligned}$$

Putting everything together, we find

$$(10.14) \lesssim_u \eta^{12} M^{-3}(1+M^3K) \lesssim_u \eta(M^{-3}+K),$$

and thus this error term is also acceptable.

We now turn to error term (10.15). By easy considerations, we only have to consider the cases $j=1$ and $j=3$. We start with the case $j=1$; using Hölder together with the Hardy–Littlewood–Sobolev inequality, Sobolev embedding, (10.7), (10.8), and (10.9), we estimate

$$\begin{aligned}
&\int_I\int_{\mathbb{R}^4}\int_{\mathbb{R}^4}\frac{|u_{hi}(t,y)|^2|u_{hi}(t,x)||u_{lo}(t,x)|^3}{|x-y|}\,dx\,dy\,dt\\
&\qquad\lesssim \|u_{hi}\|^2_{L^{12}_t L^{24/11}_x}\Big\|\tfrac{1}{|x|}*\big(|u_{hi}||u_{lo}|^3\big)\Big\|_{L^{6/5}_t L^{12}_x}\\
&\qquad\lesssim_u M^{-2}(1+M^3K)^{1/6}\|u_{hi}u_{lo}^3\|_{L^{6/5}_{t,x}}\\
&\qquad\lesssim_u M^{-2}(1+M^3K)^{1/6}\|u_{hi}\|_{L^\infty_t L^2_x}\|u_{lo}\|^3_{L^{18/5}_t L^9_x}\\
&\qquad\lesssim_u \eta^9 M^{-3}(1+M^3K).
\end{aligned}$$

Finally, to estimate the error term corresponding to $j=3$, we consider two scenarios: If $|u_{lo}| \le \delta|u_{hi}|$ for some small $\delta>0$, we absorb this contribution into the term

$$\int_I\int_{\mathbb{R}^4}\int_{\mathbb{R}^4}\frac{|u_{hi}(t,x)|^2|u_{hi}(t,y)|^4}{|x-y|}\,dx\,dy\,dt,$$

which appears in (10.11). If instead $|u_{hi}| \le \delta^{-1}|u_{lo}|$, we may estimate the contribution of this term by that of the error term corresponding to $j=1$. Thus,

$$(10.15) \lesssim_u \eta(M^{-3}+K) + \delta\int_I\int_{\mathbb{R}^4}\int_{\mathbb{R}^4}\frac{|u_{hi}(t,x)|^2|u_{hi}(t,y)|^4}{|x-y|}\,dx\,dy\,dt,$$

where $0<\delta<1$ is a constant small enough that the second term on the right-hand side above can be absorbed by (10.11). Thus, the error term (10.15) is acceptable.

We are left to consider error term (10.16). Arguing as for the case $j=1$ of the error term (10.15), we derive

$$\begin{aligned}
&\int_I\int_{\mathbb{R}^4}\int_{\mathbb{R}^4}\frac{|u_{hi}(t,y)|^2|u_{hi}(t,x)||P_{lo}F(u(t,x))|}{|x-y|}\,dx\,dy\,dt\\
&\qquad\lesssim \|u_{hi}\|^2_{L^{12}_t L^{24/11}_x}\Big\|\tfrac{1}{|x|}*\big(|u_{hi}||P_{lo}F(u)|\big)\Big\|_{L^{6/5}_t L^{12}_x}
\end{aligned}$$

$$\begin{aligned}
&\lesssim_u M^{-2}(1+M^3K)^{1/6}\|u_{hi}P_{lo}F(u)\|_{L^{6/5}_{t,x}}\\
&\lesssim_u M^{-2}(1+M^3K)^{1/6}\|u_{hi}\|_{L^4_tL^{8/3}_x}\|P_{lo}F(u)\|_{L^{12/7}_tL^{24/11}_x}\\
&\lesssim_u M^{-3}(1+M^3K)^{5/12}\|P_{lo}F(u)\|_{L^{12/7}_tL^{24/11}_x}.
\end{aligned}$$

We now write $F(u) = F(u_{hi}) + \emptyset(u_{lo}^3 + u_{lo}^2 u_{hi} + u_{lo}u_{hi}^2)$. Using Hölder, Bernstein, Sobolev embedding, (10.7), (10.8), and (10.9), we estimate

$$\begin{aligned}
\|P_{lo}\emptyset(u_{lo}^3)\|_{L^{12/7}_tL^{24/11}_x} &\lesssim \|u_{lo}\|_{L^{12}_tL^{24/5}_x}\|u_{lo}\|^2_{L^4_tL^8_x} \lesssim_u \eta^3(1+M^3K)^{7/12}\\
\|P_{lo}\emptyset(u_{lo}^2u_{hi})\|_{L^{12/7}_tL^{24/11}_x} &\lesssim M\|u_{lo}^2u_{hi}\|_{L^{12/7}_tL^{24/17}_x} \lesssim M\|u_{lo}\|^2_{L^4_tL^8_x}\|u_{hi}\|_{L^{12}_tL^{24/11}_x}\\
&\lesssim_u \eta^2(1+M^3K)^{\frac{7}{12}}\\
\|P_{lo}\emptyset(u_{lo}u_{hi}^2)\|_{L^{12/7}_tL^{24/11}_x} &\lesssim M\|u_{lo}u_{hi}^2\|_{L^{12/7}_tL^{24/17}_x}\\
&\lesssim M\|u_{lo}\|_{L^3_tL^{12}_x}\|u_{hi}\|_{L^4_tL^{8/3}_x}\|u_{hi}\|_{L^\infty_tL^4_x}\\
&\lesssim_u \eta(1+M^3K)^{7/12}.
\end{aligned}$$

Finally, using Bernstein, Hölder, interpolation, (10.7), (10.8), and (10.9), we get

$$\begin{aligned}
\|P_{lo}F(u_{hi})\|_{L^{12/7}_tL^{24/11}_x} &\lesssim M^{13/6}\|F(u_{hi})\|_{L^{12/7}_tL^1_x}\\
&\lesssim M^{13/6}\|u_{hi}\|^2_{L^{24/7}_tL^{48/17}_x}\|u_{hi}\|_{L^\infty_tL^{24/7}_x}\\
&\lesssim_u M^{1/6}(1+M^3K)^{7/12}\||\nabla|^{5/6}u_{hi}\|_{L^\infty_tL^2_x}\\
&\lesssim_u \eta(1+M^3K)^{7/12}.
\end{aligned}$$

Collecting these estimates, we find

$$(10.16) \lesssim_u \eta M^{-3}(1+M^3K) \lesssim_u \eta(M^{-3}+K),$$

and thus this last error term is also acceptable.

This completes the proof of Proposition 10.2. □

10.2. **The quasi-soliton scenario.** With Proposition 10.2 in place, we are now ready to preclude our last enemy, namely, solutions as in Theorem 8.10 for which $\int_0^{T_{max}} N(t)^{-1}\,dt = \infty$.

Theorem 10.3 (No quasi-solitons)**.** *There exist no almost periodic solutions $u : [0,T_{max})\times\mathbb{R}^4 \to \mathbb{C}$ to (8.15) with $N(t) \equiv N_k \geq 1$ on each characteristic interval $J_k \subset [0,T_{max})$ which satisfy $\|u\|_{L^6_{t,x}([0,T_{max})\times\mathbb{R}^4)} = +\infty$ and*

$$\int_0^{T_{max}} N(t)^{-1}\,dt = \infty. \tag{10.17}$$

Proof. We argue by contradiction. Assume there exists such a solution u.

Let $\eta > 0$ be a small parameter to be chosen later. By Proposition 10.2, there exists $M_0 = M_0(\eta)$ such that for all $M \leq M_0$ and any compact time interval $I \subset [0,T_{max})$, which is a union of contiguous intervals J_k, we have

$$\int_I\int_{\mathbb{R}^4}\int_{\mathbb{R}^4}\frac{|u_{\geq M}(t,x)|^2|u_{\geq M}(t,y)|^2}{|x-y|^3}\,dx\,dy\,dt \lesssim_u \eta\Big[M^{-3}+\int_I N(t)^{-1}\,dt\Big]. \tag{10.18}$$

As $\inf_{t\in[0,T_{max})} N(t) \geq 1$, choosing M_0 even smaller if necessary (depending on η) we can also ensure that

(10.19)
$$\|u_{\leq M}\|_{L_t^\infty L_x^4([0,T_{max})\times\mathbb{R}^4)} + \|u_{\leq M}\|_{L_t^\infty \dot H_x^1([0,T_{max})\times\mathbb{R}^4)} \leq \eta \quad \text{for all} \quad M \leq M_0.$$

Exercise 10.1. Use almost periodicity to prove that there exists $C(u) > 0$ such that

$$N(t)^2 \int_{|x-x(t)|\leq C(u)/N(t)} |u(t,x)|^2\, dx \gtrsim_u 1/C(u) \tag{10.20}$$

uniformly for $t \in [0, T_{max})$.

Using Hölder's inequality and (10.19), we find

$$\begin{aligned}\int_{|x-x(t)|\leq C(u)/N(t)} |u_{\leq M}(t,x)|^2\, dx &\lesssim \Big\{\frac{C(u)}{N(t)}\|u_{\leq M}\|_{L_t^\infty L_x^4([0,T_{max})\times\mathbb{R}^4)}\Big\}^2 \\ &\lesssim_u \eta^2 C(u)^2 N(t)^{-2}\end{aligned}$$

for all $t \in [0, T_{max})$ and all $M \leq M_0$. Combining this with (10.20) and choosing η sufficiently small depending on u, we find

$$\inf_{t\in[0,T_{max})} N(t)^2 \int_{|x-x(t)|\leq C(u)/N(t)} |u_{\geq M}(t,x)|^2\, dx \gtrsim_u 1 \quad \text{for all} \quad M \leq M_0.$$

Thus, on any compact time interval $I \subset [0, T_{max})$ and for any $M \leq M_0$ we have

$$\begin{aligned}&\int_I \int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \frac{|u_{\geq M}(t,x)|^2 |u_{\geq M}(t,y)|^2}{|x-y|^3}\, dx\, dy\, dt \\ &\quad\geq \int_I \iint_{|x-y|\leq \frac{2C(u)}{N(t)}} \Big[\frac{N(t)}{2C(u)}\Big]^3 |u_{\geq M}(t,x)|^2 |u_{\geq M}(t,y)|^2\, dx\, dy\, dt \\ &\quad\geq \int_I \Big[\frac{N(t)}{2C(u)}\Big]^3 \int_{|x-x(t)|\leq \frac{C(u)}{N(t)}} |u_{\geq M}(t,x)|^2\, dx \int_{|y-x(t)|\leq \frac{C(u)}{N(t)}} |u_{\geq M}(t,y)|^2\, dy\, dt \\ &\quad\gtrsim_u \int_I N(t)^{-1}\, dt.\end{aligned}$$

Invoking (10.18) and choosing η small depending on u, we find

$$\int_I N(t)^{-1}\, dt \lesssim_u M^{-3} \quad \text{for all} \quad M \leq M_0$$

and all intervals $I \subset [0, T_{max})$, which are unions of contiguous intervals J_k. Recalling the hypothesis (10.17), we derive a contradiction by choosing the interval $I \subset [0, T_{max})$ sufficiently large.

This completes the proof of the theorem. □

Appendix A. Background material

A.1. **Compactness in L^p.** Recall that by the Arzelà–Ascoli theorem, a family of continuous functions on a compact set $K \subset \mathbb{R}^d$ is precompact in $C^0(K)$ if and only if it is uniformly bounded and equicontinuous. The natural generalization to L^p spaces is due to M. Riesz [29] and reads as follows:

Proposition A.1. *Fix $1 \leq p < \infty$. A family of functions $\mathcal{F} \subset L^p(\mathbb{R}^d)$ is precompact in this topology if and only if it obeys the following three conditions:*

(i) *There exists* $A > 0$ *so that* $\|f\|_p \leq A$ *for all* $f \in \mathcal{F}$.
(ii) *For any* $\varepsilon > 0$ *there exists* $\delta > 0$ *so that* $\int_{\mathbb{R}^d} |f(x) - f(x+y)|^p \, dx < \varepsilon$ *for all* $f \in \mathcal{F}$ *and all* $|y| < \delta$.
(iii) *For any* $\varepsilon > 0$ *there exists* R *so that* $\int_{|x| \geq R} |f|^p \, dx < \varepsilon$ *for all* $f \in \mathcal{F}$.

Remark. By analogy to the case of continuous functions (or of measures) it is natural to refer to the three conditions as uniform boundedness, equicontinuity, and tightness, respectively.

Proof. If $\mathcal{F}$ is precompact, it may be covered by balls of radius $\frac{1}{2}\varepsilon$ around a finite collection of functions $\{f_j\}$. As any single function obeys (i)–(iii), these properties can be extended to the whole family by approximation by an f_j.

We now turn to sufficiency. Given $\varepsilon > 0$, our job is to show that there are finitely many functions $\{f_j\}$ so that the ε-balls centered at these points cover $\mathcal{F}$. We will find these points via the usual Arzelà–Ascoli theorem, which requires us to approximate $\mathcal{F}$ by a family of continuous functions of compact support. Let $\phi : \mathbb{R}^d \to [0, \infty)$ be a smooth function supported by $\{|x| \leq 1\}$ with $\phi(x) = 1$ in a neighbourhood of $x = 0$ and $\int_{\mathbb{R}^d} \phi(x) \, dx = 1$. Given $R > 0$ we define

$$f_R(x) := \phi\big(\tfrac{x}{R}\big) \int_{\mathbb{R}^d} R^d \phi\big(R(x-y)\big) f(y) \, dy$$

and write $\mathcal{F}_R := \{f_R : f \in \mathcal{F}\}$. Employing the three conditions, we see that it is possible to choose R so large that $\|f - f_R\|_p < \frac{1}{2}\varepsilon$ for all $f \in \mathcal{F}$. We also see that $\mathcal{F}_R$ is a uniformly bounded family of equicontinuous functions on the compact set $\{|x| \leq R\}$. Thus, $\mathcal{F}_R$ is precompact and we may find a finite family $\{f_j\} \subseteq C^0(\{|x| \leq R\})$ so that $\mathcal{F}_R$ is covered by the L^p-balls of radius $\frac{1}{2}\varepsilon$ around these points. By construction, the ε-balls around these points cover $\mathcal{F}$. $\square$

In the L^2 case it is natural to replace (ii) by a condition on the Fourier transform:

Corollary A.2. *A family of functions is precompact in* $L^2(\mathbb{R}^d)$ *if and only if it obeys the following two conditions:*
(i) *There exists* $A > 0$ *so that* $\|f\| \leq A$ *for all* $f \in \mathcal{F}$.
(ii) *For all* $\varepsilon > 0$ *there exists* $R > 0$ *so that* $\int_{|x| \geq R} |f(x)|^2 \, dx + \int_{|\xi| \geq R} |\hat{f}(\xi)|^2 \, d\xi < \varepsilon$ *for all* $f \in \mathcal{F}$.

Proof. Necessity follows as before. Regarding the sufficiency of these conditions, we note that

$$\int_{\mathbb{R}^d} |f(x+y) - f(x)|^2 \, dx \sim \int_{\mathbb{R}^d} |e^{i\xi y} - 1|^2 |\hat{f}(\xi)|^2 \, d\xi,$$

which allows us to rely on the preceding proposition. $\square$

In our applications, regularity allows us to upgrade weak-$*$ convergence to almost everywhere convergence. The lower semicontinuity of the norm under this notion of convergence is essentially Fatou's lemma. The following quantitative version of this is due to Brézis and Lieb [6] (see also [24, Theorem 1.9]):

Lemma A.3 (Refined Fatou). *Suppose* $\{f_n\} \subseteq L^p_x(\mathbb{R}^d)$ *with* $\limsup \|f_n\|_p < \infty$. *If* $f_n \to f$ *almost everywhere, then*

$$\int_{\mathbb{R}^d} \Big| |f_n|^p - |f_n - f|^p - |f|^p \Big| \, dx \to 0.$$

In particular, $\|f_n\|_p^p - \|f_n - f\|_p^p \to \|f\|_p^p$.

A.2. **Littlewood–Paley theory.** Let $\varphi(\xi)$ be a radial bump function supported in the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq \frac{11}{10}\}$ and equal to 1 on the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq 1\}$. For each number $N > 0$, we define the Fourier multipliers

$$\begin{aligned} \widehat{P_{\leq N} f}(\xi) &:= \varphi(\xi/N)\hat{f}(\xi) \\ \widehat{P_{> N} f}(\xi) &:= (1 - \varphi(\xi/N))\hat{f}(\xi) \\ \widehat{P_N f}(\xi) &:= (\varphi(\xi/N) - \varphi(2\xi/N))\hat{f}(\xi) \end{aligned}$$

and similarly $P_{<N}$ and $P_{\geq N}$. We also define

$$P_{M<\cdot\leq N} := P_{\leq N} - P_{\leq M} = \sum_{M<N'\leq N} P_{N'}$$

whenever $M < N$. We will usually use these multipliers when M and N are *dyadic numbers* (that is, of the form 2^n for some integer n); in particular, all summations over N or M are understood to be over dyadic numbers. Nevertheless, it will occasionally be convenient to allow M and N to not be a power of 2.

Like all Fourier multipliers, the Littlewood–Paley operators commute with the propagator $e^{it\Delta}$, as well as with differential operators such as $i\partial_t + \Delta$. We will use basic properties of these operators many times, including

Lemma A.4 (Bernstein estimates)**.** *For $1 \leq p \leq q \leq \infty$,*

$$\begin{aligned} \big\| |\nabla|^{\pm s} P_N f \big\|_{L^p(\mathbb{R}^d)} &\sim N^{\pm s} \|P_N f\|_{L^p(\mathbb{R}^d)}, \\ \|P_{\leq N} f\|_{L^q(\mathbb{R}^d)} &\lesssim N^{\frac{d}{p}-\frac{d}{q}} \|P_{\leq N} f\|_{L^p(\mathbb{R}^d)}, \\ \|P_N f\|_{L^q(\mathbb{R}^d)} &\lesssim N^{\frac{d}{p}-\frac{d}{q}} \|P_N f\|_{L^p(\mathbb{R}^d)}. \end{aligned}$$

Lemma A.5 (Square function estimates)**.** *Given a Schwartz function f, let*

$$S(f)(x) := \Big(\sum_N \big|P_N f(x)\big|^2\Big)^{1/2}$$

denote the Littlewood–Paley square function. For $1 < p < \infty$,

$$\|S(f)\|_{L^p(\mathbb{R}^d)} \sim \|f\|_{L^p(\mathbb{R}^d)}.$$

More generally,

$$\Big\| \Big(\sum_N N^{2s} \big|P_N f(x)\big|^2\Big)^{1/2} \Big\|_{L^p(\mathbb{R}^d)} \sim \big\| |\nabla|^s f \big\|_{L^p(\mathbb{R}^d)} \tag{A.1}$$

for all $s > -d$ and $1 < p < \infty$.

A.3. **Fractional calculus.** We first record the fractional product rule from [11]:

Lemma A.6 (Fractional product rule, [11])**.** *Let $s \in (0, 1]$ and $1 < r, p_1, p_2, q_1, q_2 < \infty$ such that $\frac{1}{r} = \frac{1}{p_i} + \frac{1}{q_i}$ for $i = 1, 2$. Then,*

$$\big\| |\nabla|^s (fg) \big\|_{L^r(\mathbb{R}^d)} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^d)} \big\| |\nabla|^s g \big\|_{L^{q_1}(\mathbb{R}^d)} + \big\| |\nabla|^s f \big\|_{L^{p_2}(\mathbb{R}^d)} \|g\|_{L^{q_2}(\mathbb{R}^d)}.$$

We will also need the following fractional chain rule from [11]. For a textbook treatment, see [37, §2.4].

Lemma A.7 (Fractional chain rule, [11])**.** *Suppose $G \in C^1(\mathbb{C})$, $s \in (0, 1]$, and $1 < p, p_1, p_2 < \infty$ are such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then,*

$$\| |\nabla|^s G(u) \|_{L^p(\mathbb{R}^d)} \lesssim \|G'(u)\|_{L^{p_1}(\mathbb{R}^d)} \| |\nabla|^s u \|_{L^{p_2}(\mathbb{R}^d)}.$$

Although we will not need it in our applications here, for completeness we record the following fractional chain rule for when the function G is no longer C^1, but merely Hölder continuous:

Lemma A.8 (Fractional chain rule for a Hölder continuous function, [39]). *Let G be a Hölder continuous function of order $0 < \alpha < 1$. Then, for every $0 < s < \alpha$, $1 < p < \infty$, and $\frac{s}{\alpha} < \sigma < 1$ we have*

$$\big\||\nabla|^s G(u)\big\|_{L^p(\mathbb{R}^d)} \lesssim \big\||u|^{\alpha-\frac{s}{\sigma}}\big\|_{L^{p_1}(\mathbb{R}^d)} \big\||\nabla|^\sigma u\big\|^{\frac{s}{\sigma}}_{L^{\frac{s}{\sigma}p_2}(\mathbb{R}^d)}, \tag{A.2}$$

provided $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $(1 - \frac{s}{\alpha\sigma})p_1 > 1$.

A.4. **A paraproduct estimate.** In Section 9, we made use of a paraproduct estimate from [40]. The proof we present here is different from the one in [40]; however, it only requires basic knowledge of harmonic analysis and so it is better suited to these lecture notes.

Lemma A.9 (Paraproduct estimate, [40]). *We have*

$$\big\||\nabla|^{-2/3}(fg)\big\|_{L^{4/3}(\mathbb{R}^4)} \lesssim \big\||\nabla|^{-2/3} f\big\|_{L^p(\mathbb{R}^4)} \big\||\nabla|^{2/3} g\big\|_{L^q(\mathbb{R}^4)},$$

for any $\frac{4}{3} < p < \infty$ and $1 < q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{11}{12}$.

Proof. The claim is equivalent to the following estimate

$$\big\||\nabla|^{-\frac{2}{3}}\{(|\nabla|^{\frac{2}{3}} f)(|\nabla|^{-\frac{2}{3}} g)\}\big\|_{L^{4/3}(\mathbb{R}^4)} \lesssim \|f\|_{L^p(\mathbb{R}^4)} \|g\|_{L^q(\mathbb{R}^4)}, \tag{A.3}$$

for $\frac{4}{3} < p < \infty$, $1 < q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{11}{12}$. To prove this, we start by performing the following decomposition:

$$\begin{aligned} |\nabla|^{-\frac{2}{3}}\{(|\nabla|^{\frac{2}{3}} f)(|\nabla|^{-\frac{2}{3}} g)\} = |\nabla|^{-\frac{2}{3}} \Big\{ &\sum_{\frac{1}{8} \le \frac{N_1}{N_2} \le 8} P_{N_1}\big(|\nabla|^{\frac{2}{3}} f\big) P_{N_2}\big(|\nabla|^{-\frac{2}{3}} g\big) \\ &+ \sum_{N_1} P_{N_1}\big(|\nabla|^{\frac{2}{3}} f\big) P_{>8N_1}\big(|\nabla|^{-\frac{2}{3}} g\big) \\ &+ \sum_{N_1} P_{N_1}\big(|\nabla|^{\frac{2}{3}} f\big) P_{<\frac{1}{8}N_1}\big(|\nabla|^{-\frac{2}{3}} g\big) \Big\}. \end{aligned} \tag{A.4}$$

Next, we will show how to control the contribution of each of the terms on the right-hand side of (A.4) to (A.3).

Using Sobolev embedding, Cauchy–Schwarz, and the square function estimate (A.1), we estimate the contribution of the first term on the right-hand side of (A.4) as follows:

$$\begin{aligned} &\Big\| |\nabla|^{-\frac{2}{3}} \sum_{\frac{1}{8} \le \frac{N_1}{N_2} \le 8} P_{N_1}\big(|\nabla|^{\frac{2}{3}} f\big) P_{N_2}\big(|\nabla|^{-\frac{2}{3}} g\big) \Big\|_{L^{4/3}} \\ &\lesssim \Big\| \sum_{\frac{1}{8} \le \frac{N_1}{N_2} \le 8} N_1^{-\frac{2}{3}} N_2^{\frac{2}{3}} \big|P_{N_1}\big(|\nabla|^{\frac{2}{3}} f\big)\big| \big|P_{N_2}\big(|\nabla|^{-\frac{2}{3}} g\big)\big| \Big\|_{L^{12/11}} \\ &\lesssim \Big\| \Big(\sum_{\frac{1}{8} \le \frac{N_1}{N_2} \le 8} \big|N_1^{-\frac{2}{3}} P_{N_1}\big(|\nabla|^{\frac{2}{3}} f\big)\big|^2 \Big)^{\frac{1}{2}} \Big(\sum_{\frac{1}{8} \le \frac{N_1}{N_2} \le 8} \big|N_2^{\frac{2}{3}} P_{N_2}\big(|\nabla|^{-\frac{2}{3}} g\big)\big|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{12/11}} \end{aligned}$$

$$\lesssim \Big\| \Big(\sum_{\frac18 \leq \frac{N_1}{N_2} \leq 8} \big| N_1^{-\frac23} P_{N_1}\big(|\nabla|^{\frac23} f\big)\big|^2 \Big)^{\frac12} \Big\|_{L^p} \Big\| \Big(\sum_{\frac18 \leq \frac{N_1}{N_2} \leq 8} \big| N_2^{\frac23} P_{N_2}\big(|\nabla|^{-\frac23} g\big)\big|^2 \Big)^{\frac12} \Big\|_{L^q}$$
$$\lesssim \|f\|_{L^p} \|g\|_{L^q}.$$

Arguing similarly, we estimate the contribution of the second term on the right-hand side of (A.4) as follows:

$$\begin{aligned}
&\Big\| |\nabla|^{-\frac23} \sum_{N_1} P_{N_1}\big(|\nabla|^{\frac23} f\big) P_{>8N_1}\big(|\nabla|^{-\frac23} g\big) \Big\|_{L^{4/3}} \\
&\qquad \lesssim \Big\| \sum_{N_1} N_1^{-\frac23} \big| P_{N_1}\big(|\nabla|^{\frac23} f\big)\big| N_1^{\frac23} \big| P_{>8N_1}\big(|\nabla|^{-\frac23} g\big)\big| \Big\|_{L^{12/11}} \\
&\qquad \lesssim \Big\| \Big(\sum_{N_1} \big| N_1^{-\frac23} P_{N_1}\big(|\nabla|^{\frac23} f\big)\big|^2 \Big)^{\frac12} \Big(\sum_{N_1} \big| N_1^{\frac23} P_{>8N_1}\big(|\nabla|^{-\frac23} g\big)\big|^2 \Big)^{\frac12} \Big\|_{L^{12/11}} \\
&\qquad \lesssim \|f\|_{L^p} \|g\|_{L^q},
\end{aligned}$$

where we also used the following consequence of (A.1):

$$\Big\| \Big(\sum_{N} N^{2s} \big| P_{\geq N} h\big|^2 \Big)^{\frac12} \Big\|_{L^p} \sim \big\| |\nabla|^s h \big\|_{L^p} \quad \text{for all} \quad s > 0 \quad \text{and} \quad 1 < p < \infty.$$

It remains to estimate the contribution of the third term on the right-hand side of (A.4). To do this, we use Lemma A.5, the easy estimates $|P_N h| \lesssim M(h)$ and $|P_{\leq N} h| \lesssim M(h)$, and the vector maximal inequality:

$$\begin{aligned}
&\Big\| |\nabla|^{-\frac23} \sum_{N_1} P_{N_1}\big(|\nabla|^{\frac23} f\big) P_{<\frac18 N_1}\big(|\nabla|^{-\frac23} g\big) \Big\|_{L^{4/3}} \\
&\qquad \lesssim \Big\| \Big(\sum_{N} \Big| N^{-\frac23} P_N \Big[\sum_{N_1} P_{N_1}\big(|\nabla|^{\frac23} f\big) P_{<\frac18 N_1}\big(|\nabla|^{-\frac23} g\big) \Big] \Big|^2 \Big)^{\frac12} \Big\|_{L^{4/3}} \\
&\qquad \lesssim \Big\| \Big(\sum_{N} \Big| N^{-\frac23} M \Big[\sum_{N_1 \sim N} P_{N_1}\big(|\nabla|^{\frac23} f\big) \Big] \Big|^2 \Big)^{\frac12} M\big(|\nabla|^{-\frac23} g\big) \Big\|_{L^{4/3}} \\
&\qquad \lesssim \Big\| \Big(\sum_{N} \sum_{N_1 \sim N} \big| N^{-\frac23} P_{N_1}\big(|\nabla|^{\frac23} f\big)\big|^2 \Big)^{\frac12} \Big\|_{L^p} \big\| M\big(|\nabla|^{-\frac23} g\big) \big\|_{L^r} \\
&\qquad \lesssim \|f\|_{L^p} \big\| |\nabla|^{-\frac23} g \big\|_{L^r},
\end{aligned}$$

where r is such that $\frac1p + \frac1r = \frac34$. (Note that this is source of the restriction $p > \frac43$.) The claim now follows by applying Sobolev embedding to the second factor on the right-hand side of the inequality above. □

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