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Nonlinear Schrödinger Equations at Critical Regularity

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1. Introduction

We will be discussing the Cauchy problem for the nonlinear Schrödinger equation:

$$\begin{cases} iu_t = -\Delta u + \mu |u|^p u \\ u(t=0,x) = u_0(x). \end{cases} \tag{1.1}$$

Here $u : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}$ is a complex-valued function of time and space, the Laplacian is in the space variables only, $\mu \in \mathbb{R}\setminus\{0\}$, and $p \geq 0$. By rescaling the values of u, it is possible to restrict attention to the cases $\mu = -1$ or $\mu = +1$; these are known as the *focusing* and *defocusing* equations, respectively.

The class of solutions to (1.1) is left invariant by the scaling

$$u(t,x) \mapsto \lambda^{\frac{2}{p}} u(\lambda^2 t, \lambda x). \tag{1.2}$$

This scaling defines a notion of criticality, specifically, for a given Banach space of initial data u_0, the problem is called *critical* if the norm is invariant under (1.2). The problem is called *subcritical* if the norm of the rescaled solution diverges as $\lambda \to \infty$; if the norm shrinks to zero, then the problem is *supercritical.* Notice that sub-/super-criticality is determined by the response of the norm to the behaviour of u_0 at small length scales, or equivalently, at high-frequencies. This is natural as the low frequencies are comparatively harmless; they are both smooth and slow-moving.

To date, most authors have focused on initial data belonging to L^2_x-based Sobolev spaces

$$\|u_0\|^2_{H^s_x} := \int_{\mathbb{R}^d} |\hat{u}_0(\xi)|^2 \,(1+|\xi|^2)^s \, d\xi \quad \text{or} \quad \|u_0\|^2_{\dot{H}^s_x} := \int_{\mathbb{R}^d} |\hat{u}_0(\xi)|^2 \,|\xi|^{2s} \, d\xi. \tag{1.3}$$

These are known as the inhomogeneous and homogeneous Sobolev spaces, respectively. The latter is better behaved under scaling, which makes it the more natural choice for studying critical problems. Let us pause to reiterate criticality in these terms.

Definition 1.1. Consider the initial value problem (1.1) for $u_0 \in \dot{H}^s_x(\mathbb{R}^d)$. This problem is *critical* when $s = s_c := \frac{d}{2} - \frac{2}{p}$, *subcritical* when $s > s_c$, and *supercritical* when $s < s_c$.

In these notes, we will be focusing on two specific critical problems, which are singled out by the fact that the critical regularity coincides with a conserved quantity. These are the *mass-critical* equation,

$$iu_t = -\Delta u + \mu |u|^{\frac{4}{d}} u, \tag{1.4}$$

which is associated with the conservation of *mass*,

$$M(u(t)) := \int_{\mathbb{R}^d} |u(t,x)|^2 \, dx, \tag{1.5}$$

and the *energy-critical* equation (in dimensions $d \geq 3$),

$$iu_t = -\Delta u + \mu |u|^{\frac{4}{d-2}} u, \tag{1.6}$$

which is associated with the conservation of *energy*,

$$E(u(t)) := \int_{\mathbb{R}^d} \tfrac{1}{2} |\nabla u(t,x)|^2 + \mu \tfrac{d-2}{2d} |u(t,x)|^{\frac{2d}{d-2}} \, dx. \tag{1.7}$$

For subcritical equations, the local problem is well understood, because it is amenable to treatment as a perturbation of the linear equation. This has lead to a satisfactory global theory at conserved regularity. A major theme of current research is to understand the global behaviour of subcritical solutions at non-conserved regularity. By comparison, supercritical equations, even at conserved regularity, are terra incognita at present.

To describe the current state of affairs regarding the mass- and energy-critical nonlinear Schrödinger equations we need to introduce a certain amount of vocabulary. We begin with what it means to be a solution of (1.4) or (1.6).

Definition 1.2 (Solution)**.** Let I be an interval containing the origin. A function $u : I \times \mathbb{R}^d \to \mathbb{C}$ is a (strong) *solution* to (1.6) if it lies in the class $C_t^0 \dot H_x^1$ and obeys the Duhamel formula

$$u(t) = e^{it\Delta}u_0 - i\mu \int_0^t e^{i(t-s)\Delta} |u(s)|^{\frac{4}{d-2}} u(s)\, ds. \tag{1.8}$$

for all $t \in I$. We say that u is a solution to (1.4) if it belongs to both $C_t^0 L_x^2$ and $L_{t,\mathrm{loc}}^{2(d+2)/d} L_x^{2(d+2)/d}$ and also obeys

$$u(t) = e^{it\Delta}u_0 - i\mu \int_0^t e^{i(t-s)\Delta} |u(s)|^{\frac{4}{d}} u(s)\, ds. \tag{1.9}$$

For the definition of $L_t^q L_x^r$ see (1.10).

When we say that (1.8) or (1.9) are obeyed, we mean as a weak integral of distributions. Note that in the mass-critical case, the nonlinearity $|u|^{\frac{4}{d}} u$ is not even a distribution for arbitrary $u \in C_t^0 L_x^2$ and $d \leq 3$. This is one reason we require u to have some additional spacetime integrability. A second reason (the primary one for $d \geq 4$) is that uniqueness of solutions is not currently known without this hypothesis. The particular spacetime integrability we require holds for solutions of the linear equation (this is Strichartz inequality, Theorem 3.2); moreover, in Section 3 we will show that (1.4) does admit local solutions in this space.

The existence of local solutions, that is, solutions on some small neighbourhood of $t = 0$, was proved by Cazenave and Weissler, [**13, 14**]. Note that in this result, the time of existence depends on the profile of u_0 rather than simply its norm. Indeed, the latter would be inconsistent with scaling invariance.

Primarily, these notes are devoted to global questions, specifically, whether the solution exists forever ($I = \mathbb{R}$) and if it does, what is its asymptotic behaviour as $t \to \pm\infty$. Here are the main notions:

Definition 1.3. A Cauchy problem is called *globally wellposed* if solutions exist for all time, are unique, and depend continuously on the initial data. A stronger notion is that the problem admits *global spacetime bounds.* In the mass-critical case, (1.4), this means that the solution u also obeys

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+2)}{d}}\, dx\, dt \leq C(M(u_0))$$

for some function C. For the analogous notion in the energy-critical case, (1.6), replace u by ∇u and u_0 by ∇u_0. We say that *asymptotic completeness* holds if for each (global) solution u there exist u_+ and u_- so that

$$u(t) - e^{it\Delta}u_+ \to 0 \quad \text{as } t \to \infty \text{ and } \quad u(t) - e^{it\Delta}u_- \to 0 \quad \text{as } t \to -\infty.$$

Note that u_+ and u_- are supposed to lie in the same space as the initial data; convergence is with respect to its norm. A converse notion is the *existence of wave operators*. This means that for each u_+ there is a global solution u of the nonlinear problem so that $u(t) - e^{it\Delta}u_+ \to 0$ and similarly for each u_-. We say *scattering* holds if wave operators exist and are asymptotically complete.

Simple arguments show that scattering follows from global spacetime bounds. In the defocusing case ($\mu = +1$), we believe that critical equations admit global spacetime bounds even when the critical Sobolev norm does not correspond to a conserved quantity. No such bold claim can hold in the focusing case; indeed, there are explicit counterexamples.

As we will discuss in Subsection 4.1, the elliptic problem

$$-\Delta f - |f|^{\frac{4}{d}} f = -f$$

on $\mathbb{R}^d$ admits Schwartz-space solutions. Indeed, there is a unique non-negative spherically symmetric Schwartz solution, which we denote by Q; see [**49, 105**]. This function is known as the *ground state*; it is, at least, the lowest eigenstate of the operator $f \mapsto -\Delta f - Q^{4/d} f$.

Now, $u(t,x) = e^{it}Q(x)$ is a global solution to the mass-critical focusing NLS that manifestly does not obey spacetime bounds, nor does it scatter (cf. (4.28)). Furthermore, by applying the pseudo-conformal identity, (2.12), we may transform this to a solution that blows up in finite time:

$$u(t,x) = (1-t)^{-\frac{d}{2}} e^{-i\frac{|x|^2}{4(1-t)} + i\frac{t}{1-t}} Q\big(\tfrac{x}{1-t}\big).$$

By comparison, the work of Cazenave and Weissler mentioned before shows that initial data of sufficiently small mass (that is, L^2_x norm) does lead to global solutions obeying spacetime bounds. Thus one may hope to identify the minimal mass at which such good behaviour first fails; $M(Q)$ is one candidate. Indeed, it is widely believed to be the correct answer:

Conjecture 1.4. *For arbitrary initial data $u_0 \in L^2_x(\mathbb{R}^d)$, the defocusing mass-critical nonlinear Schrödinger equation is globally wellposed and solutions obey global spacetime bounds; in particular, scattering holds.*

For the focusing equation, the same conclusions hold for initial data obeying $M(u_0) < M(Q)$.

Perhaps the earliest (and one of the strongest) indications that $M(Q)$ is the correct bound in the focusing case comes from work of Weinstein, [**105**], which proves global well-posedness for H^1_x initial data obeying $M(u_0) < M(Q)$. Recent progress toward settling the conjecture (at critical regularity) is discussed in the next subsection.

Before formulating the analogous conjecture for the energy-critical problem, let us discuss the natural candidate for the role of Q. By a result of Pohožaev, [**68**], the equation $-\Delta f - |f|^{\frac{4}{d-2}} f = -\beta f$ does not have $\dot H^1_x(\mathbb{R}^d)$ solutions for $\beta \neq 0$. When $\beta = 0$, this equation has a very explicit solution, namely,

$$W(x) := \big(1 + \tfrac{1}{d(d-2)}|x|^2\big)^{-\frac{d-2}{2}}.$$

From the elliptic equation, we see that $u(t,x) = W(x)$ is a stationary solution of (1.6). The general belief is that W is the minimal counterexample to global spacetime bounds in the energy-critical setting; however, the way in which it is minimal is more subtle than in the mass-critical setting. Firstly, we should not measure minimality in terms of the energy, (1.7), since the energy can be made arbitrarily negative. An alternative is to consider the kinetic energy,

$$E_0(u(t)) := \int_{\mathbb{R}^d} \tfrac{1}{2}|\nabla u(t,x)|^2 \, dx.$$

However, this creates problems of its own since it is not a conserved quantity. The solution we choose (cf. [**38, 44**]) is to assert that the only way a solution can fail to be global and obey spacetime bounds is if its kinetic energy matches (or exceeds) that of W, at least asymptotically:

Conjecture 1.5. *For arbitrary initial data $u_0 \in \dot H^1_x(\mathbb{R}^d)$, the defocusing energy-critical nonlinear Schrödinger equation is globally wellposed and solutions obey global spacetime bounds; in particular, scattering holds.*

For the focusing equation, we have the following statement: Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a solution to (1.6) *such that*

$$E_* := \sup_{t\in I} E_0(u(t)) < E_0(W).$$

Then

$$\int_I \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+2)}{d-2}} \, dx \, dt \leq C(E_*) < \infty.$$

The defocusing case of this conjecture has been completely resolved, while for the focusing equation only the three- and four-dimensional cases remain open. These results, as well as some of their precursors, are the topic of the next subsection.

1.1. Where are we? And how did we get there? We will not discuss the nonlinear wave equation in these notes; however, it seems appropriate to point out that global well-posedness for the defocusing energy-critical wave equation was proved (after considerable effort) some years before the analogous result for the nonlinear Schrödinger equation; see [**78**] where references to the original papers may be found. Treatment of the focusing energy-critical wave equation is much more recent, [**39**]. There is no analogue of mass conservation for NLW and hence no true analogue of the mass-critical NLS.

Turning now to NLS, we would like to point out two important differences between it and NLW. First, it does not enjoy finite speed of propagation. Second, in the wave case, the natural monotonicity formula (i.e., the Morawetz identity) has critical scaling; this is not the case for NLS. Both differences have had an important effect on how the theory has developed.

In [**6**], Bourgain considers the two-dimensional mass-critical NLS for inital data in L^2_x. It is shown that in order for a solution to blow up, it must concentrate some finite amount of mass in ever smaller sets (as one approaches the blowup time). Perhaps more important than the result itself were two aspects of the proof: the use of recent progress toward the restriction conjecture (see Conjecture 4.17) and a rather precise form of inverse Strichartz inequality.

Using these ingredients, Merle and Vega [**58**] obtained a concentration compactness principle for the mass-critical NLS in two dimensions. (For the analogous result in other dimensions, see [**4, 12**].) The formulation mimics results for the wave equation [**3**], although the proof is very different. The techniques used for the wave equation are better suited to the energy-critical NLS and were used by

Keraani [**41**] to obtain a concentration compactness principle for this equation. These concentration compactness principles are discussed in Section 4 and play an important role in the arguments presented in these notes. History, however, took a slightly different route.

The first major step toward verifying either conjecture was Bourgain's proof, [**7**], of global spacetime bounds for the defocusing energy-critical NLS in three and four dimensions with spherically symmetric data. A major new tool introduced therein was 'induction on energy'. We will now try to convey the outline. The role of the base step is played by the fact that global spacetime bounds are known for small data, say for data with energy less than e_0. Next we choose a small η depending on e_0. If all solutions with energy less than $e_1 := e_0+\eta$ obey satisfactory spacetime bounds then we are ready to move to the next step. Suppose not, that is, suppose that there is a (local) solution u with enormous spacetime norm, but energy less than e_1. Then, using Morawetz and inverse Strichartz-type inequalities, one may show that the there is a bubble of concentration carrying energy $\gg \eta$ that is protected by a comparatively long time interval over which u has little spacetime norm. If we remove the bubble, we obtain initial data with energy less than e_0 which then leads to a global solution with good bounds (thanks to the inductive hypothesis). Taking advantage of the buffer zone, it is possible to glue the bubble back in without completely destroying this bound. By defining what was meant earlier by 'satisfactory spacetime bound' in an appropriate manner, we reach a contradiction. This proves the result for solutions with energy less than e_1. Next, we turn our attention to solutions with energy less that $e_2 := e_1 + \eta(e_1)$, and so on, and so on.

Concentration results such as those mentioned in the previous paragraph provide important leverage in critical problems; the size of the bubbles they exhibit provide a characteristic length scale. The fact that we are dealing with scale-invariant problems means that any length scale must be dictated by the solution; it cannot be imposed from without. It is only through breaking the scaling symmetry, in a manner such as this, that non-critical tools such as the Morawetz identity can be properly brought to bear.

In [**32**], Grillakis showed global regularity for the three-dimensional energy-critical defocusing NLS with spherically symmetric initial data, that is, he proved that smooth spherically symmetric initial data leads to a global smooth solution. This can be deduced *a posteriori* from [**7**]; however, the argument in [**32**] is rather different. Subsequent progress in the spherically symmetric case, including the treatment of higher dimensions, can be found in [**89**].

The big breakthrough for non-spherically symmetric initial data was made in [**20**]. This paper brought a wealth of new ideas and tools to the problem, of which we will describe just a few. First, the authors use an interaction Morawetz inequality (introduced in [**19**]), which is much better suited to the non-symmetric case than the (Lin–Strauss) Morawetz used in previous works. See Section 7 for a discussion of both.

Unfortunately, the interaction Morawetz identity is further from critical scaling than its predecessor, which necessitates a much stronger form of concentration result. By reaping the ultimate potential of the induction on energy technique, the authors of [**20**] showed that it suffices to consider solutions that are well localized in both space and frequency. Indeed, modulo the action of scaling and space translations, these solutions remain in a very small neighbourhood of a compact set in $\dot H^1_x(\mathbb{R}^3)$.

The argument from [**20**] was generalized to four space dimensions in [**75**] and then to dimensions five and higher in [**103, 104**]. Taken together, these papers resolve the defocusing case of Conjecture 1.5.

In [**42**], Keraani used the concentration compactness statements discussed earlier to show that if the mass-critical NLS did not obey global spacetime bounds, then there is a solution u with minimal mass and infinite spacetime norm. Simple contrapositive would show that there is a sequence of global solutions with mass growing to the minimal value whose spacetime norms diverge to infinity. The point here is that the limit object exists, albeit after passing to a subsequence and performing symmetry operations. An additional immediate consequence of this compactness principle is that the minimal mass blowup solution u is almost periodic modulo symmetries (cf. Definition 5.1). This is a stronger form of concentration result than is provided by the induction on energy technique. We will turn to a more formal comparison shortly. The existence of minimal blowup solutions was adapted to the energy-critical case in [**38**], which is also the first application of this important innovation to the well-posedness problem. The main result of that paper was to prove the focusing case of Conjecture 1.5 for spherically symmetric data in dimensions $d = 3, 4, 5$. This was extended to all dimensions in [**47**]. For general (non-symmetric) data in dimensions five and higher, Conjecture 1.5 was proved in [**44**]. The complete details of this argument will be presented here. The conjecture remains open for $d = 3, 4$.

The difference between the 'minimal blowup solution' strategy and the 'induction on energy' approach is akin to that between the well ordering principle (any non-empty subset of $\{0, 1, 2, \dots\}$ contains a least element) and the principle of induction. By its intrinsically recursive nature, induction is well suited to obtaining concrete bounds and this is, indeed, what the induction on energy approach provides. By contrast, proof by contradiction, which is the basis of the minimal counterexample approach, often leads to cleaner simpler arguments, but can seldom be made effective. These general principles hold true in the NLS setting. The minimal counterexample approach leads to simpler proofs, particularly because it allows for a much more modular approach — induction on energy requires delicately interconnected arguments that cannot be disentangled until the very end — however, it does not seem possible to obtain effective bounds without reverting to the older technology. On pedagogical grounds, we will confine our attention to the minimal counterexample method in these notes.

Perhaps we have done too good a job of distinguishing the two approaches; they are two sides of the same coin: they may look very different, but are built upon the same substrate, namely, improved Strichartz inequalities. These are discussed in Subsection 4.4.

Let us now describe the current state of affairs for the mass-critical equation. Building on developments in the energy-critical case, Conjecture 1.4 has been settled for spherically symmetric data in dimensions two and higher. For the defocusing case, $d \geq 3$, see [**96, 97**]. For $d = 2$, both focusing and defocusing, see [**43**]. The latter argument was adapted to treat the $d \geq 3$ focusing case in [**46**].

With so much of the road left to travel, it would be premature to try to discern what parts of the these works may prove valuable in settling the full conjecture. We present here a number of building blocks taken from those papers that we believe will be useful in the non-symmetric case.

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1.2. Notation. We will be regularly referring to the spacetime norms

$$\|u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} := \left(\int_{\mathbb{R}} \left[\int_{\mathbb{R}^d} |u(t,x)|^r\,dx\right]^{\frac{q}{r}} dt\right)^{\frac{1}{q}}, \tag{1.10}$$

with obvious changes if q or r is infinity. To save space in in-line formulas, we will abbreviate

$$\|f\|_r := \|f\|_{L_x^r} \quad \text{and} \quad \|u\|_{q,r} := \|u\|_{L_t^q L_x^r}.$$

We write $X \lesssim Y$ to indicate that $X \leq CY$ for some constant C, which is permitted to depend on the ambient spatial dimension, d, without further comment. Other dependencies of C will be indicated with subscripts, for example, $X \lesssim_u Y$. We will write $X \sim Y$ to indicate that $X \lesssim Y \lesssim X$.

We use the 'Japanese bracket' convention: $\langle x\rangle := (1+|x|^2)^{1/2}$ as well as $\langle\nabla\rangle := (1-\Delta)^{1/2}$. Similarly, $|\nabla|^s$ denotes the Fourier multiplier with symbol $|\xi|^s$. These are used to define the Sobolev norms

$$\|f\|_{W^{s,r}} := \|\langle\nabla\rangle^s f\|_{L_x^r}.$$

Our convention for the Fourier transform is

$$\hat f(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,dx$$

so that

$$f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \hat f(\xi)\,d\xi \quad \text{and} \quad \int_{\mathbb{R}^d} |\hat f(\xi)|^2\,d\xi = \int_{\mathbb{R}^d} |f(x)|^2\,dx.$$

Notations associated to Littlewood-Paley projections are discussed in Appendix A.

2. Symmetries

2.1. Hamiltonian formulation. As we will see, the nonlinear Schrödinger equation may be viewed as an infinite dimensional Hamiltonian system. In the finite dimensional case, Hamiltonian mechanics has many general theorems of wide applicability. In the PDE setting, however, these tend to become guiding principles with each system requiring its own special treatment; indeed, compare the local theory for ODE with that for PDE. In what follows, we will take a rather formal approach, since it is not difficult to check the conclusions *a posteriori*. In particular, we will allow ourselves a rather fluid notion of phase space. In all cases, it will be a vector space of functions from $\mathbb{R}^d$ into $\mathbb{C}$. If we were working with polynomial nonlinearities, it would be reasonable to use Schwartz space. However, in the case of fractional power nonlinearities, this space is not conserved by the flow; besides, the main goal of these notes is to work in low regularity spaces.

A symplectic form is a closed non-degenerate (anti-symmetric) 2-form on phase space. In particular, it takes two tangent vectors f, g at a point u in phase space and returns a real number. The symplectic form relevant to us is

$$\omega(f,g) := \operatorname{Im} \int_{\mathbb{R}^d} f(x)\overline{g(x)}\,dx.$$

Notice that this implies $q(x) : u \mapsto \operatorname{Re} u(x)$ and $p(x) : u \mapsto \operatorname{Im} u(x)$ are canonically conjugate coordinates (indexed by x). In light of this, we see that (with the sign conventions in [**1**]) the Poisson bracket associated to ω is given by

$$\{G,F\}(u) = \int_{\mathbb{R}^d} \frac{\delta F}{\delta p}\Big|_u(x)\frac{\delta G}{\delta q}\Big|_u(x) - \frac{\delta F}{\delta q}\Big|_u(x)\frac{\delta G}{\delta p}\Big|_u(x)\,dx, \tag{2.1}$$

where the functional derivatives are defined by

$$\lim_{\varepsilon\to 0}\frac{G(u+\varepsilon v)-G(u)}{\varepsilon} = dG\big|_u(v) = \int_{\mathbb{R}^d} \frac{\delta G}{\delta q}\Big|_u(x)\operatorname{Re} v(x) + \frac{\delta G}{\delta p}\Big|_u(x)\operatorname{Im} v(x)\,dx$$

for all $v : \mathbb{R}^d \to \mathbb{C}$. In particular, $\{q(y), p(x)\}(u) = \delta(x-y)$, independent of u, which expresses the fact that these are canonically conjugate coordinates.

For a general real-valued function H defined on phase space, the associated (Hamiltonian) flow is defined by

$$u_t = \nabla_\omega H(u) \quad \text{where the vector field } \nabla_\omega H \text{ is defined by} \quad dH(\cdot) = \omega(\cdot, \nabla_\omega H).$$

A consequence (or alternate definition) is that for any function F on phase space,

$$\tfrac{d}{dt}F(u(t)) = \{F,H\}(u(t)).$$

In particular, $q_t = \frac{\delta H}{\delta p}$ and $p_t = -\frac{\delta H}{\delta q}$, which are the usual form of Hamilton's equations. When needed, we will write $\exp(t\nabla_\omega H)$ for the time-t flow map.

With all these notions in place, we leave the final (indeed central) point to the reader:

Exercise. Show that formally, the Hamiltonian

$$H(u) := \int_{\mathbb{R}^d} \tfrac{1}{2}|\nabla u|^2 + \tfrac{\mu}{p+2}|u|^{p+2}\,dx \tag{2.2}$$

leads to the flow

$$iu_t = -\Delta u + \mu|u|^p u. \tag{2.3}$$

2.2. The symmetries. In this subsection, we will list the main symmetries of (2.3), together with a brief discussion of each.

Recall that Noether's Theorem guarantees that there is a bijection between conserved quantities and one-parameter groups of symplectomorphisms preserving the Hamiltonian. Specifically, using the conserved quantity as a Hamiltonian leads to a (symplectic form preserving) flow that conserves the original Hamiltonian. In each case that this theorem is applicable, we will note the corresponding conservation law. Some important symmetries do not preserve the symplectic form and/or the Hamiltonian; nevertheless, we will still be able to find an appropriate substitute for a corresponding conserved quantity.

Time translations. If $u(t)$ is a solution of (2.3), then clearly so is $u(t+\tau)$ for τ fixed. This symmetry is associated with conservation of the Hamiltonian (2.2).

Space translations. It is not difficult to see that both the Hamiltonian (2.2) and the symplectic/Poisson structure are invariant under spatial translations: $u(t,x) \mapsto u(t, x - x_0)$. This symmetry is generated by the total momentum

$$P(u) := \int_{\mathbb{R}^d} 2\,\mathrm{Im}\big(\bar{u}\nabla u\big)\,dx. \tag{2.4}$$

Indeed, given $x_0 \in \mathbb{R}^d$,

$$u(x - x_0) = \big[e^{\frac{1}{4}\nabla_\omega (x_0\cdot P)}u\big](x).$$

The factor 2 has been included in (2.4) to match conventions elsewhere.

Space rotations. Invariance under rotations of the coordinate axes corresponds to the conservation of angular momentum. The later is a tensor with $\binom{d}{2}$ components, indexed by pairs $1 \leq j < k \leq d$:

$$L_{jk}(u) = i\int_{\mathbb{R}^d} \bar{u}[x_j\partial_k u - x_k\partial_j u]\,dx.$$

Concomitant with the non-commutativity of the rotation group $SO(d)$, the components of angular momentum do not all Poisson commute with one another, forming instead, a representation of the Lie algebra $so(d)$.

Phase rotations. The map $u(x) \mapsto e^{i\theta}u(x)$ is a simple form of gauge symmetry. It is connected to the conservation of mass:

$$M(u) := \int_{\mathbb{R}^d} |u|^2\,dx \quad \text{obeys} \quad e^{\tau\nabla_\omega M}u = e^{-2i\tau}u. \tag{2.5}$$

Time reversal. As intuition dictates, one may invert the time evolution by simply reversing all momenta. Given our choice of canonical coordinates, this corresponds to the map $u \mapsto \bar{u}$. We leave the reader to check that

$$e^{t\nabla_\omega H}\bar{u} = \overline{e^{-t\nabla_\omega H}u}.$$

Galilei boosts. A central tenet of mechanics is that the same laws of motion apply in all inertial (non-accelerating) reference frames. Combined with an absolute notion of time, this leads directly to Galilean relativity.

The class of solutions to the nonlinear Schrödinger equation (2.3) is left invariant by Galilei boosts:

$$u(t,x) \mapsto e^{ix\cdot\xi_0 - it|\xi_0|^2}u(t, x - 2\xi_0 t), \tag{2.6}$$

where $\xi_0 \in \mathbb{R}^d$ denotes (half the) relative velocity of the two reference frames.

There are two (connected) problems with applying Noether's Theorem in this case: the symmetry explicitly involves time, it is not simply a transformation of phase space, and it does not leave the Hamiltonian invariant (cf. Proposition 2.3 below). As we will explain, the appropriate substitute for a conserved quantity is

$$X(u) := \int_{\mathbb{R}^d} x|u|^2\,dx. \tag{2.7}$$

This represents the location of the centre of mass, at least when $M(u) = 1$.

The time derivative of X is

$$\{X, H\} = P, \quad \text{which implies} \quad \{\{X, H\}, H\} = 0. \tag{2.8}$$

Thus, although it is not conserved, X has a very simple time evolution:

$$X(u(t)) = X(u(0)) + t\cdot P(u(0)).$$

It remains for us to connect X with Galilei boosts. The first indication of this is

$$\big[e^{-\frac{1}{2}\nabla_\omega(\xi_0\cdot X)}u\big](x)=e^{ix\cdot\xi_0}u(x),$$

which reproduces the action of a Galilei boost on the initial data $u(t=0)$. Perhaps this is enough to convince the reader of a connection; however, we wish to use this example to elucidate a little abstract theory. The central tenet is quite simple: One may extend the privileged status of conserved quantities, that is, those obeying $\{F,H\}=0$, to those functions F that together with H generate a finite-dimensional Lie algebra under the action of the Poisson bracket. The concomitant group multiplication law gives a form of time-dependent symmetry.

Together with the Hamiltonian, X generates a $(2d+2)$-dimensional Lie algebra under the action of the Poisson bracket. The basis vectors are H, M, and $X_j,\ P_j$, $1\leq j\leq d$ and the only non-zero brackets among them are

$$\{X,H\}=P \quad\text{and}\quad \{X_j,P_k\}=4\delta_{jk}M. \tag{2.9}$$

Note that (X,P,M) form the Heisenberg Lie algebra; indeed, the corresponding flows (on u) exactly reproduce the standard Schrödinger representation of the Heisenberg group. Using the (Lie group) commutation laws induced by (2.9), we obtain

$$e^{t\nabla_\omega H}e^{-\frac{1}{2}\nabla_\omega(\xi_0\cdot X)}=e^{\frac{t}{2}\nabla_\omega(\xi_0\cdot P-|\xi_0|^2M)}e^{-\frac{1}{2}\nabla_\omega(\xi_0\cdot X)}e^{t\nabla_\omega H},$$

which is exactly the statement that (2.6) preserves solutions to (2.3).

Scaling. The scaling symmetry for (2.3) is

$$u(t,x)\mapsto\lambda^{\frac{2}{p}}u(\lambda^2t,\lambda x). \tag{2.10}$$

This does not preserve the symplectic/Poisson structure, except in the mass-critical ($p=\frac{4}{d}$) case. It does not preserve the Hamiltonian unless $p=\frac{4}{d-2}$, which corresponds to the energy-critical equation.

As noted, the mass-critical scaling does preserve the symplectic/Poisson structure, which guarantees that it is generated by some Hamiltonian flow. A few computations reveal that

$$A(u):=\tfrac{1}{4i}\int_{\mathbb{R}^d}\bar u(x\cdot\nabla+\nabla\cdot x)u\,dx=\tfrac12\int_{\mathbb{R}^d}x\cdot\operatorname{Im}(\bar u\nabla u)\,dx$$

obeys

$$\big[e^{-\tau\nabla_\omega A}u\big](x)=e^{\frac{d}{2}\tau}u(e^\tau x).$$

and further, that

$$\{A,H\}=2H+\tfrac{\mu(pd-4)}{2(p+2)}\int_{\mathbb{R}^d}|u|^{2+p}\,dx.$$

This is the best substitute we have for a conservation law associated to (2.10). The peculiar combination of kinetic and potential energies on the right-hand side actually turns out to play an important role; see Section 7.

Specializing to the mass-critical or the linear Schrödinger equation, we obtain the simple relation $\{A,H\}=2H$, which is much more amenable to a Lie-theoretic perspective. In particular,

$$e^{t\nabla_\omega H}e^{-\tau\nabla_\omega A}=e^{-\tau\nabla_\omega A}e^{e^{2\tau}t\nabla_\omega H},$$

which reproduces (2.10).

Lens transformations. An idealized lens advances (or retards) the phase of the incident wave in proportion to the square of the distance to the optical axis. This leads us to consider

$$V(u) := \int_{\mathbb{R}^d} |x|^2 |u|^2 \, dx, \tag{2.11}$$

which is the generator of lens transformations:

$$[e^{\tau \nabla_\omega V} u](x) = e^{-2i\tau|x|^2} u(x).$$

The time evolution of V is given by $\{V, H\} = 8A$.

Under the linear or mass-critical nonlinear Schrödinger evolutions, A behaves in a simple manner, as we discussed above. This leads directly to a time-dependent symmetry, known as the *pseudo-conformal symmetry*; see (2.12) below. We leave the computations to the reader's private pleasure:

Exercise. In the mass-critical (or linear) case, H, A, V form a three dimensional Lie algebra with relations $\{A, H\} = 2H$, $\{V, H\} = 8A$, and $\{V, A\} = 2V$. By comparing this with matrices of the form

$$\begin{bmatrix} -a & -8v \\ h & a \end{bmatrix},$$

show that this is the Lie algebra of $SL_2(\mathbb{R})$. Use this (or not) to verify that

$$\left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right] : \psi(t, x) \mapsto (\beta t + \delta)^{-\frac{d}{2}} e^{\frac{i\beta|x|^2}{4(\beta t+\delta)}} \psi\big(\tfrac{\alpha t+\gamma}{\beta t+\delta}, \tfrac{x}{\beta t+\delta}\big) \tag{2.12}$$

gives an explicit representation of $SL_2(\mathbb{R})$ on the class of mass-critical solutions.

2.3. Group therapy. The main purpose of this subsection is to introduce some notation we will be using for (a subgroup of) the symmetries just introduced. After that, we will record the effect of symmetries on the major conserved quantities.

Definition 2.1 (Mass-critical symmetry group)**.** For any phase $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, position $x_0 \in \mathbb{R}^d$, frequency $\xi_0 \in \mathbb{R}^d$, and scaling parameter $\lambda > 0$, we define the unitary transformation $g_{\theta, x_0, \xi_0, \lambda} : L^2_x(\mathbb{R}^d) \to L^2_x(\mathbb{R}^d)$ by the formula

$$[g_{\theta, \xi_0, x_0, \lambda} f](x) := \frac{1}{\lambda^{d/2}} e^{i\theta} e^{ix\cdot\xi_0} f\big(\frac{x - x_0}{\lambda}\big).$$

We let G be the collection of such transformations. If $u : I \times \mathbb{R}^d \to \mathbb{C}$, we define $T_{g_{\theta,\xi_0,x_0,\lambda}} u : \lambda^2 I \times \mathbb{R}^d \to \mathbb{C}$, where $\lambda^2 I := \{\lambda^2 t : t \in I\}$, by the formula

$$[T_{g_{\theta,\xi_0,x_0,\lambda}} u](t, x) := \frac{1}{\lambda^{d/2}} e^{i\theta} e^{ix\cdot\xi_0} e^{-it|\xi_0|^2} u\left(\frac{t}{\lambda^2}, \frac{x - x_0 - 2\xi_0 t}{\lambda}\right),$$

or equivalently,

$$[T_{g_{\theta,\xi_0,x_0,\lambda}} u](t) = g_{\theta - t|\xi_0|^2, \xi_0, x_0 + 2\xi_0 t, \lambda}\Big(u\big(\lambda^{-2} t\big)\Big).$$

Note that if u is a solution to the mass-critical NLS, then $T_g u$ is also solution and has initial data $g[u(t = 0)]$.

Definition 2.2 (Energy-critical symmetry group)**.** For any phase $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, position $x_0 \in \mathbb{R}^d$, and scaling parameter $\lambda > 0$, we define a unitary transformation $g_{\theta, x_0, \lambda} : \dot{H}^1_x(\mathbb{R}^d) \to \dot{H}^1_x(\mathbb{R}^d)$ by

$$[g_{\theta, x_0, \lambda} f](x) := \lambda^{-\frac{d-2}{2}} e^{i\theta} f\big(\lambda^{-1}(x - x_0)\big).$$

Let G denote the collection of such transformations. For a function $u : I \times \mathbb{R}^d \to \mathbb{C}$, we define $T_{g_{\theta,x_0,\lambda}} u : \lambda^2 I \times \mathbb{R}^d \to \mathbb{C}$, where $\lambda^2 I := \{\lambda^2 t : t \in I\}$, by the formula

$$[T_{g_{\theta,x_0,\lambda}} u](t,x) := \lambda^{-\frac{d-2}{2}} e^{i\theta} u\big(\lambda^{-2}t, \lambda^{-1}(x - x_0)\big).$$

Note that if u is a solution to the energy-critical NLS, then so is $T_g u$; the latter has initial data $g[u(t=0)]$.

The next proposition shows how the total mass, momentum, and energy are affected by elements of the mass- or energy-critical symmetry groups. In the latter case, we also record the effect of Galilei boosts. Although they have been omitted from the definition of the symmetry group (they will not be required in the concentration compactness step), they are valuable in further simplifying the structure of minimal blowup solutions.

Proposition 2.3 (Mass, Momentum, and Energy under symmetries)**.** *Let g be an element of the mass-critical symmetry group with parameters θ, x, ξ, and λ. Then*

$$\begin{aligned} M(gu_0) &= M(u_0), \quad P(gu_0) = 2\xi M(u) + \lambda^{-1} P(u_0), \\ E(gu_0) &= \lambda^{-2} E(u_0) + \tfrac{1}{2}\lambda^{-1}\xi \cdot P(u_0) + \tfrac{1}{2}|\xi|^2 M(u_0). \end{aligned} \tag{2.13}$$

The analogous statement for the energy-critical case reads

$$\begin{aligned} M(v_0) &= \lambda^2 M(u_0), \quad P(v_0) = 2\lambda^2 \xi M(u_0) + \lambda P(u_0), \\ E(v_0) &= E(u_0) + \tfrac{1}{2}\lambda\xi \cdot P(u_0) + \tfrac{1}{2}\lambda^2|\xi|^2 M(u_0), \end{aligned} \tag{2.14}$$

where $v_0(x) = [e^{-\frac{1}{2}\nabla_\omega(\xi\cdot X)} g u_0](x) = e^{ix\cdot\xi}[gu_0](x)$.

Corollary 2.4 (Minimal energy in the rest frame)**.** *Let $\tilde{u} \in L_t^\infty H_x^1$ be a blowup solution to the mass- or energy-critical NLS. Then there is a blowup solution $u \in L_t^\infty H_x^1$, obeying $M(u) = M(\tilde{u})$, $E(u) \leq E(\tilde{u})$, and*

$$P(u(t)) = 2 \operatorname{Im} \int_{\mathbb{R}^d} \overline{u(t,x)} \nabla u(t,x)\, dx \equiv 0.$$

Note also that $\|\nabla u\|_{\infty,2} \leq \|\nabla \tilde{u}\|_{\infty,2}$.

Proof. Choose u to be the unique Galilei boost of $\tilde{u}$ that has zero momentum. All the conclusions now follow quickly from the formulae above. Note that u has minimal energy among all Galilei boosts of $\tilde{u}$; indeed, this is an expression of the well-know physical fact that the total energy can be decomposed as the energy viewed in the centre of mass frame plus the energy arising from the motion of the center of mass (cf. [**50**, §8]). □

2.4. Complete integrability. The purpose of this subsection is to share an observation of Jürgen Moser: scattering implies complete integrability. This was passed on to us by Percy Deift.

In the finite dimensional setting, a Hamiltonian flow on a $2n$-dimensional phase space is called *completely integrable* if it admits n functionally independent Poisson commuting conserved quantities. An essentially equivalent formulation is the existence of action-angle coordinates (cf. [**1**]). These are a system of canonically conjugate coordinates $I_1, \ldots, I_n, \phi_1, \ldots, \phi_n$, which is to say

$$\{I_j, I_k\} = \{\phi_j, \phi_k\} = 0 \qquad \{I_j, \phi_k\} = \delta_{jk},$$

so that under the flow,

$$\tfrac{d}{dt} I_j = 0 \quad \text{and} \quad \tfrac{d}{dt}\phi_j = \omega_j(I_1, \ldots, I_n).$$

Here $\omega_1, \ldots, \omega_n$ are smooth functions.

In what follows, we will exemplify Moser's assertion in the context of the mass-critical defocusing equation. For clarity of exposition, we presuppose the truth of the associated global well-posedness and scattering conjecture. The principal ideas can be applied to any NLS setting.

As we will see in Section 3, we are guaranteed that the wave operator

$$\Omega : u_0 \mapsto u_+ = \lim_{t\to\infty} e^{-it\Delta} u(t)$$

defines a bijection on $L_x^2(\mathbb{R}^d)$; here $u(t)$ denotes the solution of NLS with initial data u_0. In fact, since both the free Schrödinger and the NLS evolutions are Hamiltonian, the wave operator preserves the symplectic form. As the Fourier transform is also bijective and symplectic (both follow from unitarity), so is the combined map

$$\hat\Omega : u_0 \mapsto \widehat{u_+}, \quad \text{which obeys} \quad \big[\hat\Omega(u(t))\big](\xi) = e^{-it|\xi|^2}\widehat{u_+}(\xi).$$

Thus we have found a symplectic map that trivializes the flow; moreover, we have an infinite family of Poisson commuting conserved quantities, namely,

$$u \mapsto \int_{\mathbb{R}^d} g(\xi) |\widehat{u_+}(\xi)|\, d\xi$$

as g varies over real-valued functions in $L_\xi^2(\mathbb{R}^d)$. Lastly, to see that these do indeed Poisson commute and also to exhibit action-angle variables, we note that if we define $I(\xi) = \frac12 |\widehat{u_+}(\xi)|^2$ and $\phi(\xi)$ by $\widehat{u_+}(\xi) = |\widehat{u_+}(\xi)| e^{-i\phi(\xi)}$, then

$$\{I(\xi), I(\eta)\} = \{\phi(\xi), \phi(\eta)\} = 0, \quad \{I(\xi), \phi(\eta)\} = \delta(\xi - \eta),$$
$$\tfrac{d}{dt} I(\xi) = 0, \quad \text{and} \quad \tfrac{d}{dt}\phi(\xi) = |\xi|^2.$$

Remark. By integrating $|\widehat{u_+}(\xi)|^2$ against appropriate powers of ξ, one obtains conserved quantities that agree with the asymptotic $\dot H_x^s$ norm. For $s = 0$ or $s = 1$, these are exactly the mass and energy. For general values of s, the conserved quantities need not take such a simple (polynomial in u, $\bar u$, and their derivatives) form.

3. The local theory

3.1. Dispersive and Strichartz inequalities. It is not difficult to check (or derive) that the fundamental solution of the heat equation is given by

$$e^{s\Delta}(x,y) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y) - s|\xi|^2}\, d\xi = (4\pi s)^{-d/2} e^{-|x-y|^2/4s}$$

for all $s > 0$. By analytic continuation, we find the fundamental solution of the free Schrödinger equation:

$$e^{it\Delta}(x,y) = (4\pi i t)^{-d/2} e^{i|x-y|^2/4t} \tag{3.1}$$

for all $t \neq 0$. Note that here

$$(4\pi i t)^{-d/2} = (4\pi |t|)^{-d/2} e^{-i\pi d\, \operatorname{sign}(t)/4}.$$

From (3.1) one easily derives the standard dispersive inequality

$$\|e^{it\Delta} f\|_{L_x^p(\mathbb{R}^d)} \lesssim |t|^{d(\frac1p - \frac12)} \|f\|_{L_x^{p'}(\mathbb{R}^d)} \tag{3.2}$$

for all $t \neq 0$ and $2 \leq p \leq \infty$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

A different way to express the dispersive effect of the operator $e^{it\Delta}$ is in terms of spacetime integrability. To state the estimates, we first need the following definition.

Definition 3.1 (Admissible pairs)**.** For $d \geq 1$, we say that a pair of exponents (q, r) is *Schrödinger-admissible* if

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad 2 \leq q, r \leq \infty, \quad \text{and} \quad (d, q, r) \neq (2, 2, \infty). \tag{3.3}$$

For a fixed spacetime slab $I \times \mathbb{R}^d$, we define the *Strichartz norm*

$$\|u\|_{S^0(I)} := \sup_{(q,r) \text{ admissible}} \|u\|_{L_t^q L_x^r(I \times \mathbb{R}^d)} \tag{3.4}$$

We write $S^0(I)$ for the closure of all test functions under this norm and denote by $N^0(I)$ the dual of $S^0(I)$.

Remark. In the case of two space dimensions, the absence of the endpoint requires us to restrict the supremum in (3.4) to a closed subset of admissible pairs. As any reasonable argument only involves finitely many admissible pairs, this is of little consequence.

We are now ready to state the standard Strichartz estimates:

Theorem 3.2 (Strichartz)**.** *Let $0 \leq s \leq 1$, let I be a compact time interval, and let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a solution to the forced Schrödinger equation*

$$iu_t + \Delta u = F.$$

Then,

$$\| |\nabla|^s u\|_{S^0(I)} \lesssim \|u(t_0)\|_{\dot{H}_x^s} + \| |\nabla|^s F\|_{N^0(I)}$$

for any $t_0 \in I$.

Proof. We will treat the non-endpoint cases in Subsection 4.4 following [**28, 83**]. For the endpoint $(q, r) = \big(2, \frac{2d}{d-2}\big)$ in dimensions $d \geq 3$, see [**37**]. For failure of the $d = 2$ endpoint, see [**59**]. This endpoint can be partially recovered in the case of spherically symmetric functions; see [**82, 87**]. □

3.2. The $\dot{H}_x^s$ critical case. In this subsection we revisit the local theory at critical regularity. Consider the initial-value problem

$$\begin{cases} iu_t + \Delta u = F(u) \\ u(0) = u_0 \end{cases} \tag{3.5}$$

where $u(t, x)$ is a complex-valued function of spacetime $\mathbb{R} \times \mathbb{R}^d$ with $d \geq 1$. Assume that the nonlinearity $F : \mathbb{C} \to \mathbb{C}$ is continuously differentiable and obeys the power-type estimates

$$F(z) = O\big(|z|^{1+p}\big) \tag{3.6}$$

$$F_z(z),\ F_{\bar{z}}(z) = O\big(|z|^p\big) \tag{3.7}$$

$$F_z(z) - F_z(w),\ F_{\bar{z}}(z) - F_{\bar{z}}(w) = O\big(|z - w|^{\min\{p,1\}}(|z| + |w|)^{\max\{0,p-1\}}\big) \tag{3.8}$$

for some $p > 0$, where F_z and $F_{\bar{z}}$ are the usual complex derivatives

$$F_z := \frac{1}{2}\Big(\frac{\partial F}{\partial x} - i\frac{\partial F}{\partial y}\Big), \quad F_{\bar{z}} := \frac{1}{2}\Big(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\Big).$$

For future reference, we record the chain rule

$$\nabla F(u(x)) = F_z(u(x))\nabla u(x) + F_{\bar z}(u(x))\overline{\nabla u(x)}, \tag{3.9}$$

as well as the closely related integral identity

$$F(z)-F(w) = (z-w)\int_0^1 F_z\big(w+\theta(z-w)\big)\,d\theta + \overline{(z-w)}\int_0^1 F_{\bar z}\big(w+\theta(z-w)\big)\,d\theta \tag{3.10}$$

for any $z, w \in \mathbb{C}$; in particular, from (3.7), (3.10), and the triangle inequality, we have the estimate

$$\big|F(z) - F(w)\big| \lesssim |z-w|\big(|z|^p + |w|^p\big). \tag{3.11}$$

The model example of a nonlinearity obeying the conditions above is $F(u) = |u|^p u$, for which the critical homogeneous Sobolev space is $\dot H^{s_c}_x$ with $s_c := \frac{d}{2} - \frac{2}{p}$. The local theory for (3.5) at this critical regularity was developed by Cazenave and Weissler [**13, 14, 15**]. Like them, we are interested in strong solutions to (3.5).

Definition 3.3 (Solution)**.** A function $u : I\times\mathbb{R}^d \to \mathbb{C}$ on a non-empty time interval $0 \in I \subset \mathbb{R}$ is a *solution* (more precisely, a strong $\dot H^{s_c}_x(\mathbb{R}^d)$ solution) to (3.5) if it lies in the class $C^0_t \dot H^{s_c}_x(K\times\mathbb{R}^d) \cap L^{p+2}_t L_x^{\frac{dp(p+2)}{4}}(K\times\mathbb{R}^d)$ for all compact $K \subset I$, and obeys the Duhamel formula

$$u(t) = e^{it\Delta}u(0) - i\int_0^t e^{i(t-s)\Delta}F(u(s))\,ds \tag{3.12}$$

for all $t \in I$. We refer to the interval I as the *lifespan* of u. We say that u is a *maximal-lifespan solution* if the solution cannot be extended to any strictly larger interval. We say that u is a *global solution* if $I = \mathbb{R}$.

Note that for $s_c \in \{0, 1\}$, this is slightly different from the definition of solution given in the introduction. However, one of the consequences of the theory developed in this section is that the two notions are equivalent.

Theorem 3.4 (Standard local well-posedness, [**13, 14, 15**])**.** *Let $d \ge 1$ and $u_0 \in H^{s_c}_x(\mathbb{R}^d)$. Assume further that $0 \le s_c \le 1$. There exists $\eta_0 = \eta_0(d) > 0$ such that if $0 < \eta \le \eta_0$ and I is a compact interval containing zero such that*

$$\big\||\nabla|^{s_c} e^{it\Delta}u_0\big\|_{L^{p+2}_t L_x^{\frac{2d(p+2)}{2(d-2)+dp}}(I\times\mathbb{R}^d)} \le \eta, \tag{3.13}$$

then there exists a unique solution u to (3.5) on $I\times\mathbb{R}^d$. Moreover, we have the bounds

$$\big\||\nabla|^{s_c} u\big\|_{L^{p+2}_t L_x^{\frac{2d(p+2)}{2(d-2)+dp}}(I\times\mathbb{R}^d)} \le 2\eta \tag{3.14}$$

$$\big\||\nabla|^{s_c} u\big\|_{S^0(I\times\mathbb{R}^d)} \lesssim \big\||\nabla|^{s_c} u_0\big\|_{L^2_x} + \eta^{1+p} \tag{3.15}$$

$$\|u\|_{S^0(I\times\mathbb{R}^d)} \lesssim \|u_0\|_{L^2_x}. \tag{3.16}$$

Remarks. 1. By Strichartz inequality, we know that

$$\big\||\nabla|^{s_c} e^{it\Delta}u_0\big\|_{L^{p+2}_t L_x^{\frac{2d(p+2)}{2(d-2)+dp}}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \big\||\nabla|^{s_c}u_0\big\|_{L^2_x}.$$

Thus, (3.13) holds for initial data with sufficiently small norm. Alternatively, by the monotone convergence theorem, (3.13) holds provided I is chosen sufficiently small.

Note that by scaling, the length of the interval I depends on the fine properties of u_0, not only on its norm.

2. Note that the initial data in the theorem above is assumed to belong to the *inhomogeneous* Sobolev space $H_x^{s_c}(\mathbb{R}^d)$, as in the work of Cazenave and Weissler. This makes the proof significantly simpler. In the next two subsections, we will present a technique which allows one to show uniform continuous dependence of the solution u upon the initial data u_0 in *critical* spaces. This technique (or indeed, the result) can be used to treat initial data in the *homogeneous* Sobolev space $\dot{H}_x^{s_c}(\mathbb{R}^d)$.

3. The sole purpose of the restriction to $s_c \leq 1$ is to simplify the statement and proof. In any event, it covers the two cases of greatest interest to us, $s_c = 0, 1$.

PROOF. We will essentially repeat the original argument from [**14**]; the fractional chain rule Lemma A.11 leads to some simplifications.

The theorem follows from a contraction mapping argument. More precisely, using the Strichartz estimates from Theorem 3.2, we will show that the solution map $u \mapsto \Phi(u)$ defined by

$$\Phi(u)(t) := e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}F(u(s))\,ds,$$

is a contraction on the set $B_1 \cap B_2$ where

$$B_1 := \Big\{u \in L_t^\infty H_x^{s_c}(I\times\mathbb{R}^d) : \|u\|_{L_t^\infty H_x^{s_c}(I\times\mathbb{R}^d)} \leq 2\|u_0\|_{H_x^{s_c}} + C(d)(2\eta)^{1+p}\Big\}$$

$$B_2 := \Big\{u \in L_t^{p+2}W_x^{s_c,\frac{2d(p+2)}{2(d-2)+dp}}(I\times\mathbb{R}^d) : \big\||\nabla|^{s_c}u\big\|_{L_t^{p+2}L_x^{\frac{2d(p+2)}{2(d-2)+dp}}(I\times\mathbb{R}^d)} \leq 2\eta$$

$$\text{and} \quad \big\|u\big\|_{L_t^{p+2}L_x^{\frac{2d(p+2)}{2(d-2)+dp}}(I\times\mathbb{R}^d)} \leq 2C(d)\|u_0\|_{L_x^2}\Big\}$$

under the metric given by

$$d(u,v) := \|u-v\|_{L_t^{p+2}L_x^{\frac{2d(p+2)}{2(d-2)+dp}}(I\times\mathbb{R}^d)}.$$

Here $C(d)$ denotes the constant from the Strichartz inequality. Note that the norm appearing in the metric scales like L_x^2; see the second remark above. Note that both B_1 and B_2 are closed (and hence complete) in this metric.

Using Strichartz inequality followed by the fractional chain rule Lemma A.11 and Sobolev embedding, we find that for $u \in B_1 \cap B_2$,

$$\begin{aligned}
\|\Phi(u)\|&_{L_t^\infty H_x^{s_c}(I\times\mathbb{R}^d)}\\
&\leq \|u_0\|_{H_x^{s_c}} + C(d)\big\|\langle\nabla\rangle^{s_c}F(u)\big\|_{L_t^{\frac{p+2}{p+1}}L_x^{\frac{2d(p+2)}{2(d+2)+dp}}(I\times\mathbb{R}^d)}\\
&\leq \|u_0\|_{H_x^{s_c}} + C(d)\big\|\langle\nabla\rangle^{s_c}u\big\|_{L_t^{p+2}L_x^{\frac{2d(p+2)}{2(d-2)+dp}}(I\times\mathbb{R}^d)}\big\|u\big\|^p_{L_t^{p+2}L_x^{\frac{dp(p+2)}{4}}(I\times\mathbb{R}^d)}\\
&\leq \|u_0\|_{H_x^{s_c}} + C(d)\big(2\eta + 2C(d)\|u_0\|_{L_x^2}\big)\big\||\nabla|^{s_c}u\big\|^p_{L_t^{p+2}L_x^{\frac{2d(p+2)}{2(d-2)+dp}}(I\times\mathbb{R}^d)}\\
&\leq \|u_0\|_{H_x^{s_c}} + C(d)\big(2\eta + 2C(d)\|u_0\|_{L_x^2}\big)(2\eta)^p
\end{aligned}$$

and similarly,

$$\begin{aligned}
\big\|\Phi(u)\big\|_{L_t^{p+2}L_x^{\frac{2d(p+2)}{2(d-2)+dp}}(I\times\mathbb{R}^d)} &\leq C(d)\|u_0\|_{L_x^2} + C(d)\big\|F(u)\big\|_{L_t^{\frac{p+2}{p+1}}L_x^{\frac{2d(p+2)}{2(d+2)+dp}}(I\times\mathbb{R}^d)}\\
&\leq C(d)\|u_0\|_{L_x^2} + 2C(d)^2\|u_0\|_{L_x^2}(2\eta)^p.
\end{aligned}$$

Arguing as above and invoking (3.13), we obtain

$$\begin{aligned}\big\||\nabla|^{s_c}\Phi(u)\big\|_{L_t^{p+2}L_x^{\frac{2d(p+2)}{2(d-2)+dp}}(I\times\mathbb{R}^d)} &\leq \eta + C(d)\big\||\nabla|^{s_c}F(u)\big\|_{L_t^{\frac{p+2}{p+1}}L_x^{\frac{2d(p+2)}{2(d+2)+dp}}(I\times\mathbb{R}^d)}\\ &\leq \eta + C(d)(2\eta)^{1+p}.\end{aligned}$$

Thus, choosing $\eta_0 = \eta_0(d)$ sufficiently small, we see that for $0 < \eta \leq \eta_0$, the functional Φ maps the set $B_1 \cap B_2$ back to itself. To see that Φ is a contraction, we repeat the computations above and use (3.11) to obtain

$$\begin{aligned}\big\|\Phi(u)-\Phi(v)\big\|_{L_t^{p+2}L_x^{\frac{2d(p+2)}{2(d-2)+dp}}(I\times\mathbb{R}^d)} &\leq C(d)\big\|F(u)-F(v)\big\|_{L_t^{\frac{p+2}{p+1}}L_x^{\frac{2d(p+2)}{2(d+2)+dp}}(I\times\mathbb{R}^d)}\\ &\leq C(d)(2\eta)^p\|u-v\|_{L_t^{p+2}L_x^{\frac{2d(p+2)}{2(d-2)+dp}}(I\times\mathbb{R}^d)}.\end{aligned}$$

Thus, choosing $\eta_0 = \eta_0(d)$ even smaller (if necessary), we can guarantee that Φ is a contraction on the set $B_1 \cap B_2$. By the contraction mapping theorem, it follows that Φ has a fixed point in $B_1 \cap B_2$. Moreover, noting that Φ maps into $C_t^0 H_x^{s_c}$ (not just $L_t^\infty H_x^{s_c}$), we derive that the fixed point of Φ is indeed a solution to (3.5).

We now turn our attention to the uniqueness statement. Since uniqueness is a local property, it suffices to study a neighbourhood of $t = 0$. By Definition 3.3, any solution to (3.5) belongs to $B_1 \cap B_2$ on some such neighbourhood. Uniqueness thus follows from uniqueness in the contraction mapping theorem.

The claims (3.15) and (3.16) follow from another application of Strichartz inequality, as above. □

We end this section with a collection of statements which encapsulate the local theory for (3.5).

Corollary 3.5 (Local theory, [**13, 14, 15**])**.** *Let $d \geq 1$ and $u_0 \in H_x^{s_c}(\mathbb{R}^d)$. Assume also that $0 \leq s_c \leq 1$. Then there exists a unique maximal-lifespan solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ to (3.5) with initial data $u(0) = u_0$. This solution also has the following properties:*

- *(Local existence) I is an open neighbourhood of zero.*
- *(Energy and mass conservation) The mass of u is conserved, that is, $M(u(t)) = M(u_0)$ for all $t \in I$. Moreover, if $s_c = 1$ then the energy of u is also conserved, that is, $E(u(t)) = E(u_0)$ for all $t \in I$.*
- *(Blowup criterion) If $\sup I$ is finite, then u blows up forward in time, that is, there exists a time $t \in I$ such that*

$$\big\|u\big\|_{L_t^{p+2}L_x^{\frac{pd(p+2)}{4}}([t,\sup I)\times\mathbb{R}^d)} = \infty.$$

A similar statement holds in the negative time direction.

- *(Scattering) If $\sup I = +\infty$ and u does not blow up forward in time, then u scatters forward in time, that is, there exists a unique $u_+ \in H_x^{s_c}(\mathbb{R}^d)$ such that*

$$\lim_{t\to+\infty}\|u(t) - e^{it\Delta}u_+\|_{H_x^{s_c}(\mathbb{R}^d)} = 0. \tag{3.17}$$

Conversely, given $u_+ \in H_x^{s_c}(\mathbb{R}^d)$ there exists a unique solution to (3.5) in a neighbourhood of infinity so that (3.17) holds.

- *(Small data global existence) If $\big\||\nabla|^{s_c}u_0\big\|_2$ is sufficiently small (depending on d), then u is a global solution which does not blow up either forward or backward* *in time. Indeed,*

$$\big\||\nabla|^{s_c} u\big\|_{S^0(\mathbb{R})} \lesssim \big\||\nabla|^{s_c} u_0\big\|_2. \tag{3.18}$$

• *(Unconditional uniqueness in the energy-critical case) Suppose* $s_c = 1$ *and* $\tilde u \in C_t^0 \dot H_x^1(J \times \mathbb{R}^d)$ *obeys* (3.12) *and* $\tilde u(t_0) = u_0$, *then* $J \subseteq I$ *and* $\tilde u \equiv u$ *throughout* J.

PROOF. The corollary is a consequence of Theorem 3.4 and its proof. We leave it as an exercise. □

3.3. Stability: the mass-critical case. An important part of the local well-posedness theory is the study of how the strong solutions built in the previous subsection depend upon the initial data. More precisely, we would like to know whether small perturbations of the initial data lead to small changes in the solution. More generally, we are interested in developing a *stability* theory for (3.5). By stability, we mean the following property: Given an *approximate* solution to (3.5), say $\tilde u$ obeying

$$\begin{cases} i\tilde u_t + \Delta \tilde u = F(\tilde u) + e \\ \tilde u(0,x) = \tilde u_0(x) \end{cases}$$

with e small in a suitable space and $\tilde u_0 - u_0$ small in $\dot H_x^{s_c}$, then there exists a *genuine* solution u to (3.5) which stays very close to $\tilde u$ in critical norms. The question of continuous dependence of the solution upon the initial data corresponds to taking $e = 0$; the case where $e \neq 0$ can be used to consider situations where NLS is only an approximate model for the physical system under consideration.

Although stability is a local question, it plays an important role in all existing treatments of the global well-posedness problem for NLS at critical regularity. It has also proved useful in the treatment of local and global questions for more exotic nonlinearities [**95, 108**].

In these notes, we will only address the stability question for the mass- and energy-critical NLS. The techniques we will employ (particularly, those from the next subsection) can be used to develop a stability theory for the more general equation (3.5). We start with the mass-critical equation, which is the more elementary of the two. That is to say, for the remainder of this subsection we adopt the following

Convention. The nonlinearity F obeys (3.6) through (3.8) and (3.11) with $p = 4/d$.

Lemma 3.6 (Short-time perturbations, [**95**])**.** *Let* I *be a compact interval and let* $\tilde u$ *be an approximate solution to* (3.5) *in the sense that*

$$(i\partial_t + \Delta)\tilde u = F(\tilde u) + e,$$

for some function e. *Assume that*

$$\|\tilde u\|_{L_t^\infty L_x^2(I\times\mathbb{R}^d)} \leq M \tag{3.19}$$

for some positive constant M. *Let* $t_0 \in I$ *and let* $u(t_0)$ *be such that*

$$\|u(t_0) - \tilde u(t_0)\|_{L_x^2} \leq M' \tag{3.20}$$

for some $M' > 0$. *Assume also the smallness conditions*

$$\|\tilde u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)} \leq \varepsilon_0 \tag{3.21}$$

$$\big\| e^{i(t-t_0)\Delta}\big(u(t_0)-\tilde{u}(t_0)\big)\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)} \le \varepsilon \tag{3.22}$$

$$\|e\|_{N^0(I)} \le \varepsilon, \tag{3.23}$$

for some $0<\varepsilon\le\varepsilon_0$ *where* $\varepsilon_0=\varepsilon_0(M,M')>0$ *is a small constant. Then, there exists a solution* u *to* (3.5) *on* $I\times\mathbb{R}^d$ *with initial data* $u(t_0)$ *at time* $t=t_0$ *satisfying*

$$\|u-\tilde{u}\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)} \lesssim \varepsilon \tag{3.24}$$

$$\|u-\tilde{u}\|_{S^0(I)} \lesssim M' \tag{3.25}$$

$$\|u\|_{S^0(I)} \lesssim M+M' \tag{3.26}$$

$$\|F(u)-F(\tilde{u})\|_{N^0(I)} \lesssim \varepsilon. \tag{3.27}$$

Remark. Note that by Strichartz,

$$\big\| e^{i(t-t_0)\Delta}\big(u(t_0)-\tilde{u}(t_0)\big)\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)} \lesssim \|u(t_0)-\tilde{u}(t_0)\|_{L_x^2},$$

so hypothesis (3.22) is redundant if $M'=O(\varepsilon)$.

Proof. By symmetry, we may assume $t_0=\inf I$. Let $w:=u-\tilde{u}$. Then w satisfies the following initial value problem

$$\begin{cases} iw_t+\Delta w = F(\tilde{u}+w)-F(\tilde{u})-e \\ w(t_0)=u(t_0)-\tilde{u}(t_0). \end{cases}$$

For $t\in I$ we define

$$A(t):=\big\|F(\tilde{u}+w)-F(\tilde{u})\big\|_{N^0([t_0,t])}.$$

By (3.21),

$$\begin{aligned} A(t) &\lesssim \big\|F(\tilde{u}+w)-F(\tilde{u})\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([t_0,t]\times\mathbb{R}^d)} \\ &\lesssim \|w\|^{1+\frac{4}{d}}_{L_{t,x}^{\frac{2(d+2)}{d}}([t_0,t]\times\mathbb{R}^d)} + \|\tilde{u}\|^{\frac{4}{d}}_{L_{t,x}^{\frac{2(d+2)}{d}}([t_0,t]\times\mathbb{R}^d)} \|w\|_{L_{t,x}^{\frac{2(d+2)}{d}}([t_0,t]\times\mathbb{R}^d)} \\ &\lesssim \|w\|^{1+\frac{4}{d}}_{L_{t,x}^{\frac{2(d+2)}{d}}([t_0,t]\times\mathbb{R}^d)} + \varepsilon_0^{\frac{4}{d}} \|w\|_{L_{t,x}^{\frac{2(d+2)}{d}}([t_0,t]\times\mathbb{R}^d)}. \end{aligned} \tag{3.28}$$

On the other hand, by Strichartz, (3.22), and (3.23), we get

$$\begin{aligned} \|w\|_{L_{t,x}^{\frac{2(d+2)}{d}}([t_0,t]\times\mathbb{R}^d)} &\lesssim \|e^{i(t-t_0)\Delta}w(t_0)\|_{L_{t,x}^{\frac{2(d+2)}{d}}([t_0,t]\times\mathbb{R}^d)} + A(t) + \|e\|_{N^0([t_0,t])} \\ &\lesssim A(t)+\varepsilon. \end{aligned} \tag{3.29}$$

Combining (3.28) and (3.29), we obtain

$$A(t) \lesssim (A(t)+\varepsilon)^{1+\frac{4}{d}} + \varepsilon_0^{\frac{4}{d}}(A(t)+\varepsilon).$$

A standard continuity argument then shows that if ε_0 is taken sufficiently small,

$$A(t)\lesssim\varepsilon \quad \text{for any} \quad t\in I,$$

which implies (3.27). Using (3.27) and (3.29), one easily derives (3.24). Moreover, by Strichartz, (3.20), (3.23), and (3.27),

$$\|w\|_{S^0(I)} \lesssim \|w(t_0)\|_{L_x^2} + \|F(\tilde{u}+w)-F(\tilde{u})\|_{N^0(I)} + \|e\|_{N^0(I)} \lesssim M'+\varepsilon,$$

which establishes (3.25) for $\varepsilon_0=\varepsilon_0(M')$ sufficiently small.

To prove (3.26), we use Strichartz, (3.19), (3.20), (3.27), and (3.21):

$$\begin{aligned}\|u\|_{S^0(I)} &\lesssim \|u(t_0)\|_{L^2_x} + \|F(u)\|_{N^0(I)} \\ &\lesssim \|\tilde u(t_0)\|_{L^2_x} + \|u(t_0)-\tilde u(t_0)\|_{L^2_x} + \|F(u)-F(\tilde u)\|_{N^0(I)} + \|F(\tilde u)\|_{N^0(I)} \\ &\lesssim M + M' + \varepsilon + \|\tilde u\|^{1+\frac4d}_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)} \\ &\lesssim M + M' + \varepsilon + \varepsilon_0^{1+\frac4d}.\end{aligned}$$

Choosing $\varepsilon_0 = \varepsilon_0(M, M')$ sufficiently small, this finishes the proof of the lemma. □

Building upon the previous result, we are now able to prove stability for the mass-critical NLS.

Theorem 3.7 (Mass-critical stability result, [**95**]). *Let I be a compact interval and let $\tilde u$ be an approximate solution to* (3.5) *in the sense that*

$$(i\partial_t + \Delta)\tilde u = F(\tilde u) + e,$$

for some function e. Assume that

$$\|\tilde u\|_{L_t^\infty L_x^2(I\times\mathbb{R}^d)} \le M \tag{3.30}$$

$$\|\tilde u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)} \le L, \tag{3.31}$$

for some positive constants M and L. Let $t_0 \in I$ and let $u(t_0)$ obey

$$\|u(t_0) - \tilde u(t_0)\|_{L^2_x} \le M' \tag{3.32}$$

for some $M' > 0$. Moreover, assume the smallness conditions

$$\big\|e^{i(t-t_0)\Delta}\big(u(t_0) - \tilde u(t_0)\big)\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)} \le \varepsilon \tag{3.33}$$

$$\|e\|_{N^0(I)} \le \varepsilon, \tag{3.34}$$

for some $0 < \varepsilon \le \varepsilon_1$ where $\varepsilon_1 = \varepsilon_1(M, M', L) > 0$ is a small constant. Then, there exists a solution u to (3.5) *on $I \times \mathbb{R}^d$ with initial data $u(t_0)$ at time $t = t_0$ satisfying*

$$\|u - \tilde u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)} \le \varepsilon C(M, M', L) \tag{3.35}$$

$$\|u - \tilde u\|_{S^0(I)} \le C(M, M', L)M' \tag{3.36}$$

$$\|u\|_{S^0(I)} \le C(M, M', L). \tag{3.37}$$

Proof. Subdivide I into $J \sim (1 + \frac{L}{\varepsilon_0})^{\frac{2(d+2)}{d}}$ subintervals $I_j = [t_j, t_{j+1}]$, $0 \le j < J$, such that

$$\|\tilde u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_j\times\mathbb{R}^d)} \le \varepsilon_0,$$

where $\varepsilon_0 = \varepsilon_0(M, 2M')$ is as in Lemma 3.6. We need to replace M' by $2M'$ as the mass of the difference $u - \tilde u$ might grow slightly in time.

By choosing ε_1 sufficiently small depending on J, M, and M', we can apply Lemma 3.6 to obtain for each j and all $0 < \varepsilon < \varepsilon_1$

$$\begin{aligned}\|u - \tilde u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_j\times\mathbb{R}^d)} &\le C(j)\varepsilon \\ \|u - \tilde u\|_{S^0(I_j)} &\le C(j)M' \\ \|u\|_{S^0(I_j)} &\le C(j)(M + M')\end{aligned}$$

$$\|F(u) - F(\tilde{u})\|_{N^0(I_j)} \le C(j)\varepsilon,$$

provided we can prove that analogues of (3.32) and (3.33) hold with t_0 replaced by t_j. In order to verify this, we use an inductive argument. By Strichartz, (3.32), (3.34), and the inductive hypothesis,

$$\begin{aligned}\|u(t_j) - \tilde{u}(t_j)\|_{L_x^2} &\lesssim \|u(t_0) - \tilde{u}(t_0)\|_{L_x^2} + \|F(u) - F(\tilde{u})\|_{N^0([t_0,t_j])} + \|e\|_{N^0([t_0,t_j])} \\ &\lesssim M' + \sum_{k=0}^{j-1} C(k)\varepsilon + \varepsilon.\end{aligned}$$

Similarly, by Strichartz, (3.33), (3.34), and the inductive hypothesis,

$$\begin{aligned}&\big\|e^{i(t-t_j)\Delta}\big(u(t_j) - \tilde{u}(t_j)\big)\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_j\times\mathbb{R}^d)} \\ &\lesssim \big\|e^{i(t-t_0)\Delta}\big(u(t_0) - \tilde{u}(t_0)\big)\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_j\times\mathbb{R}^d)} + \|e\|_{N^0([t_0,t_j])} \\ &\qquad + \|F(u) - F(\tilde{u})\|_{N^0([t_0,t_j])} \\ &\lesssim \varepsilon + \sum_{k=0}^{j-1} C(k)\varepsilon.\end{aligned}$$

Choosing ε_1 sufficiently small depending on J, M, and M', we can guarantee that the hypotheses of Lemma 3.6 continue to hold as j varies. □

3.4. Stability: the energy-critical case. In this subsection we address the stability theory for the energy-critical NLS, that is, we adopt the following

Convention. The nonlinearity F obeys (3.6) through (3.8) and (3.11) with $p = 4/(d-2)$ and $d \ge 3$.

To motivate the approach we will take, let us consider the question of continuous dependence of the solution upon the initial data. To make things as simple as possible, let us choose initial data $u_0, \tilde{u}_0 \in H_x^1$ which are small:

$$\|u_0\|_{\dot{H}_x^1} + \|\tilde{u}_0\|_{\dot{H}_x^1} \le \eta_0.$$

By Corollary 3.5, if η_0 is sufficiently small, there exist unique global solutions u and $\tilde{u}$ to (3.5) with initial data u_0 and $\tilde{u}_0$, respectively; moreover, they satisfy

$$\|\nabla u\|_{S^0(\mathbb{R})} + \|\nabla \tilde{u}\|_{S^0(\mathbb{R})} \lesssim \eta_0.$$

We would like to see that if u_0 and $\tilde{u}_0$ are *close* in $\dot{H}_x^1$, say $\|\nabla(u_0 - \tilde{u}_0)\|_2 \le \varepsilon \ll \eta_0$, then u and $\tilde{u}$ remain close in *energy-critical* norms, measured in terms of ε, not η_0. An application of Strichartz inequality combined with the bounds above yields

$$\|\nabla(u - \tilde{u})\|_{S^0(\mathbb{R})} \lesssim \|\nabla(u_0 - \tilde{u}_0)\|_{L_x^2} + \eta_0^{\frac{4}{d-2}} \|\nabla(u - \tilde{u})\|_{S^0(\mathbb{R})} + \eta_0 \|\nabla(u - \tilde{u})\|_{S^0(\mathbb{R})}^{\frac{4}{d-2}}.$$

If $4/(d-2) \ge 1$, a simple bootstrap argument will imply continuous dependence of the solution upon the initial data. However, this will not work if $4/(d-2) < 1$, that is, if $d > 6$. The obstacle comes from the last term above; tiny numbers become much larger when raised to a fractional power. Ultimately, the problem stems from the fact that in high dimensions the derivative maps F_z and $F_{\bar{z}}$ are merely Hölder continuous rather than Lipschitz. The remedy is to work in spaces with fractional derivatives (rather than a full derivative), while still maintaining criticality with respect to the scaling. This is the approach taken by Tao and Visan

[**94**], who proved stability for the energy-critical NLS in all dimensions $d \geq 3$ (see also [**20, 75**] for earlier treatments in dimensions $d = 3, 4$). A similar technique was employed by Nakanishi [**64**] for the energy-critical Klein-Gordon equation in high dimensions.

Here we present a small improvement upon the results obtained in [**94**] made possible by the fractional chain rule for fractional powers; see Lemma A.12. The proof is rather involved and will occupy the remainder of this subsection. It is joint work with Xiaoyi Zhang (unpublished).

Theorem 3.8 (Energy-critical stability result)**.** *Let I be a compact time interval and let $\tilde u$ be an approximate solution to* (3.5) *on $I \times \mathbb{R}^d$ in the sense that*

$$i\tilde u_t + \Delta \tilde u = F(\tilde u) + e$$

for some function e. Assume that

$$\|\tilde u\|_{L_t^\infty \dot H_x^1(I\times\mathbb{R}^d)} \leq E \tag{3.38}$$

$$\|\tilde u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I\times\mathbb{R}^d)} \leq L \tag{3.39}$$

for some positive constants E and L. Let $t_0 \in I$ and let $u(t_0)$ obey

$$\|u(t_0) - \tilde u(t_0)\|_{\dot H_x^1} \leq E' \tag{3.40}$$

for some positive constant E'. Assume also the smallness conditions

$$\big\|e^{i(t-t_0)\Delta}\big(u(t_0) - \tilde u(t_0)\big)\big\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I\times\mathbb{R}^d)} \leq \varepsilon \tag{3.41}$$

$$\|\nabla e\|_{N^0(I)} \leq \varepsilon \tag{3.42}$$

for some $0 < \varepsilon < \varepsilon_1 = \varepsilon_1(E, E', L)$. Then, there exists a unique strong solution $u : I \times \mathbb{R}^d \mapsto \mathbb{C}$ to (3.5) *with initial data $u(t_0)$ at time $t = t_0$ satisfying*

$$\|u - \tilde u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I\times\mathbb{R}^d)} \lesssim C(E, E', L)\varepsilon^c \tag{3.43}$$

$$\|\nabla(u - \tilde u)\|_{\dot S^0(I)} \lesssim C(E, E', L)\, E' \tag{3.44}$$

$$\|\nabla u\|_{\dot S^0(I)} \lesssim C(E, E', L), \tag{3.45}$$

where $0 < c = c(d) < 1$.

Remark. The result in [**94**] assumes

$$\Big(\sum_{N\in 2^{\mathbb{Z}}} \big\|\nabla P_N e^{i(t-t_0)\Delta}\big(u(t_0) - \tilde u(t_0)\big)\big\|^2_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}(I\times\mathbb{R}^d)}\Big)^{1/2} \leq \varepsilon$$

in place of (3.41). Note that by Sobolev embedding, this is a strictly stronger requirement.

One of the consequences of the theorem above is a local well-posedness statement in energy-critical norms. More precisely, in Theorem 3.4 and Corollary 3.5 one can remove the assumption that the initial data belongs to L_x^2, since every $\dot H_x^1$ function is well approximated by H_x^1 functions. Alternatively, one may use the techniques we present to prove the following corollary directly. The approach we have chosen is motivated by the desire to introduce the difficulties one at a time.

Corollary 3.9 (Local well-posedness)**.** *Let I be a compact time interval, $t_0 \in I$, and let $u_0 \in \dot H^1_x(\mathbb{R}^d)$. Assume that*

$$\|u_0\|_{\dot H^1_x} \leq E.$$

Then for any $\varepsilon > 0$ there exists $\delta = \delta(E, \varepsilon) > 0$ such that if

$$\big\|e^{i(t-t_0)\Delta} u_0\big\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I\times\mathbb{R}^d)} < \delta,$$

then there exists a unique solution u to (3.5) with initial data u_0 at time $t = t_0$. Moreover,

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I\times\mathbb{R}^d)} \leq \varepsilon \quad \textit{and} \quad \|\nabla u\|_{S^0(I)} \leq 2E.$$

We now turn our attention to the proof of Theorem 3.8. Let us first introduce the spaces we will use; as mentioned above, these are critical with respect to scaling and have a small fractional number of derivatives. Throughout the remainder of this subsection, for any time interval I we will use the abbreviations

$$\begin{aligned}
\|u\|_{X^0(I)} &:= \|u\|_{L_t^{\frac{d(d+2)}{2(d-2)}} L_x^{\frac{2d^2(d+2)}{(d+4)(d-2)^2}}(I\times\mathbb{R}^d)} \\
\|u\|_{X(I)} &:= \big\||\nabla|^{\frac{4}{d+2}} u\big\|_{L_t^{\frac{d(d+2)}{2(d-2)}} L_x^{\frac{2d^2(d+2)}{d^3-4d+16}}(I\times\mathbb{R}^d)} \qquad (3.46)\\
\|F\|_{Y(I)} &:= \big\||\nabla|^{\frac{4}{d+2}} F\big\|_{L_t^{\frac{d}{2}} L_x^{\frac{2d^2(d+2)}{d^3+4d^2+4d-16}}(I\times\mathbb{R}^d)}.
\end{aligned}$$

First, we connect the spaces in which the solution to (3.5) is measured to the spaces in which the nonlinearity is measured. As usual, this is done via a Strichartz inequality; we reproduce the standard proof.

Lemma 3.10 (Strichartz estimate)**.** *Let I be a compact time interval containing t_0. Then*

$$\Big\|\int_{t_0}^t e^{i(t-s)\Delta} F(s)\, ds\Big\|_{X(I)} \lesssim \|F\|_{Y(I)}.$$

Proof. By the dispersive estimate (3.2),

$$\big\|e^{i(t-s)\Delta} F(s)\big\|_{L_x^{\frac{2d^2(d+2)}{d^3-4d+16}}} \lesssim |t-s|^{-\frac{d^2+2d-8}{d(d+2)}} \|F(s)\|_{L_x^{\frac{2d^2(d+2)}{d^3+4d^2+4d-16}}}.$$

An application of the Hardy-Littlewood-Sobolev inequality yields

$$\Big\|\int_{t_0}^t e^{i(t-s)\Delta} F(s) ds\Big\|_{L_t^{\frac{d(d+2)}{2(d-2)}} L_x^{\frac{2d^2(d+2)}{d^3-4d+16}}(I\times\mathbb{R}^d)} \lesssim \|F\|_{L_t^{\frac{d}{2}} L_x^{\frac{2d^2(d+2)}{d^3+4d^2+4d-16}}(I\times\mathbb{R}^d)}.$$

As the differentiation operator $|\nabla|^{\frac{4}{d+2}}$ commutes with the free evolution, we recover the claim. □

We next establish some connections between the spaces defined in (3.46) and the usual Strichartz spaces.

Lemma 3.11 (Interpolations)**.** *For any compact time interval I,*

$$\|u\|_{X^0(I)} \lesssim \|u\|_{X(I)} \lesssim \|\nabla u\|_{S^0(I)} \qquad (3.47)$$

$$\|u\|_{X(I)} \lesssim \|u\|^{\frac{1}{d+2}}_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I\times\mathbb{R}^d)} \|\nabla u\|^{\frac{d+1}{d+2}}_{S^0(I)} \qquad (3.48)$$

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I\times\mathbb{R}^d)} \lesssim \|u\|_{X(I)}^c \|\nabla u\|_{S^0(I)}^{1-c}, \tag{3.49}$$

where $0 < c = c(d) \leq 1$.

PROOF. A simple application of Sobolev embedding yields (3.47).

Using interpolation followed by Sobolev embedding,

$$\begin{aligned}\|u\|_{X(I)} &\lesssim \|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I\times\mathbb{R}^d)}^{\frac{1}{d+2}} \big\||\nabla|^{\frac{4}{d+1}} u\big\|_{L_t^{\frac{2d(d+1)(d+2)}{(d-2)(3d+8)}} L_x^{\frac{2d^2(d+1)(d+2)}{d^4+d^3-2d^2+8d+32}}(I\times\mathbb{R}^d)}^{\frac{d+1}{d+2}} \\ &\lesssim \|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I\times\mathbb{R}^d)}^{\frac{1}{d+2}} \|\nabla u\|_{S^0(I)}^{\frac{d+1}{d+2}}.\end{aligned}$$

This settles (3.48).

To establish (3.49), we analyze two cases. When $d = 3$, interpolation yields

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I\times\mathbb{R}^d)} \lesssim \|u\|_{X^0(I)}^{\frac{3}{4}} \|u\|_{L_t^\infty L_x^{\frac{2d}{d-2}}(I\times\mathbb{R}^d)}^{\frac{1}{4}}$$

and the claim follows (with $c = \frac{3}{4}$) from (3.47) and Sobolev embedding. For $d \geq 4$, another application of interpolation gives

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I\times\mathbb{R}^d)} \lesssim \|u\|_{X^0(I)}^{\frac{2}{d-2}} \|u\|_{L_t^{\frac{2d}{d-2}} L_x^{\frac{2d^2}{(d-2)^2}}(I\times\mathbb{R}^d)}^{\frac{d-4}{d-2}}$$

and the claim follows again (with $c = \frac{2}{d-2}$) from (3.47) and Sobolev embedding. □

Finally, we derive estimates that will help us control the nonlinearity. The main tools we use in deriving these estimates are the fractional chain rules; see Lemmas A.11 and A.12.

Lemma 3.12 (Nonlinear estimates). *Let I a compact time interval. Then,*

$$\|F(u)\|_{Y(I)} \lesssim \|u\|_{X(I)}^{\frac{d+2}{d-2}} \tag{3.50}$$

and

(3.51)

$$\begin{aligned}&\|F_z(u+v)w\|_{Y(I)} + \|F_{\bar z}(u+v)\bar w\|_{Y(I)} \\ &\qquad\lesssim \Big(\|u\|_{X(I)}^{\frac{8}{d^2-4}} \|\nabla u\|_{S^0(I)}^{\frac{4d}{d^2-4}} + \|v\|_{X(I)}^{\frac{8}{d^2-4}} \|\nabla v\|_{S^0(I)}^{\frac{4d}{d^2-4}}\Big)\|w\|_{X(I)}.\end{aligned}$$

PROOF. Throughout the proof, all spacetime norms are on $I \times \mathbb{R}^d$.

Applying Lemma A.11 combined with (3.7) and (3.47) we find

$$\|F(u)\|_{Y(I)} \lesssim \|u\|_{L_t^{\frac{d(d+2)}{2(d-2)}} L_x^{\frac{2d^2(d+2)}{(d-2)^2(d+4)}}}^{\frac{4}{d-2}} \big\||\nabla|^{\frac{4}{d+2}} u\big\|_{L_t^{\frac{d(d+2)}{2(d-2)}} L_x^{\frac{2d^2(d+2)}{d^3-4d+16}}} \lesssim \|u\|_{X(I)}^{\frac{d+2}{d-2}}.$$

This establishes (3.50).

We now turn to (3.51); we only treat the first term on the left-hand side, as the second can be handled similarly. By Lemma A.10 followed by (3.7) and (3.47),

$$\begin{aligned}&\|F_z(u+v)w\|_{Y(I)} \\ &\qquad\lesssim \|F_z(u+v)\|_{L_t^{\frac{d(d+2)}{8}} L_x^{\frac{d^2(d+2)}{2(d-2)(d+4)}}} \big\||\nabla|^{\frac{4}{d+2}} w\big\|_{L_t^{\frac{d(d+2)}{2(d-2)}} L_x^{\frac{2d^2(d+2)}{d^3-4d+16}}}\end{aligned}$$

$$+ \big\| |\nabla|^{\frac{4}{d+2}} F_z(u+v) \big\|_{L_t^{\frac{d(d+2)}{8}} L_x^{\frac{d^2(d+2)}{2d^2+8d-16}}} \|w\|_{X^0(I)}$$

$$\lesssim \|u+v\|_{X^0(I)}^{\frac{4}{d-2}} \|w\|_{X(I)} + \big\| |\nabla|^{\frac{4}{d+2}} F_z(u+v) \big\|_{L_t^{\frac{d(d+2)}{8}} L_x^{\frac{d^2(d+2)}{2d^2+8d-16}}} \|w\|_{X(I)}.$$

Thus, the claim will follow from (3.47), once we establish

$$\big\| |\nabla|^{\frac{4}{d+2}} F_z(u+v) \big\|_{L_t^{\frac{d(d+2)}{8}} L_x^{\frac{d^2(d+2)}{2d^2+8d-16}}} \lesssim \|u\|_{X(I)}^{\frac{8}{d^2-4}} \|\nabla u\|_{S^0(I)}^{\frac{4d}{d^2-4}} + \|v\|_{X(I)}^{\frac{8}{d^2-4}} \|\nabla v\|_{S^0(I)}^{\frac{4d}{d^2-4}}. \tag{3.52}$$

In dimensions $3 \le d \le 5$, this follows from Lemma A.11 and (3.47):

$$\big\| |\nabla|^{\frac{4}{d+2}} F_z(u+v) \big\|_{L_t^{\frac{d(d+2)}{8}} L_x^{\frac{d^2(d+2)}{2d^2+8d-16}}} \lesssim \|u+v\|_{X^0(I)}^{\frac{6-d}{d-2}} \|u+v\|_{X(I)} \lesssim \|u+v\|_{X(I)}^{\frac{4}{d-2}}.$$

To derive (3.52) in dimensions $d \ge 6$, we apply Lemma A.12 (with $\alpha := \frac{4}{d-2}$, $s := \frac{4}{d+2}$, and $\sigma := \frac{d}{d+2}$) followed by Hölder's inequality in the time variable, Sobolev embedding, and (3.47):

$$\begin{aligned}
&\big\| |\nabla|^{\frac{4}{d+2}} F_z(u+v) \big\|_{L_t^{\frac{d(d+2)}{8}} L_x^{\frac{d^2(d+2)}{2d^2+8d-16}}} \\
&\quad\lesssim \|u+v\|_{L_t^{\frac{d(d+2)}{2(d-2)}} L_x^{\frac{2d^2(d+2)}{(d+4)(d-2)^2}}}^{\frac{8}{d(d-2)}} \big\| |\nabla|^{\frac{d}{d+2}} (u+v) \big\|_{L_t^{\frac{d(d+2)}{2(d-2)}} L_x^{\frac{2d^2(d+2)}{d^3+2d^2-12d+16}}}^{\frac{4}{d}} \\
&\quad\lesssim \big\| |\nabla|^{\frac{d}{d+2}} (u+v) \big\|_{L_t^{\frac{d(d+2)}{2(d-2)}} L_x^{\frac{2d^2(d+2)}{d^3+2d^2-12d+16}}}^{\frac{4}{d-2}} \\
&\quad\lesssim \|u\|_{X(I)}^{\frac{8}{d^2-4}} \|\nabla u\|_{S^0(I)}^{\frac{4d}{d^2-4}} + \|v\|_{X(I)}^{\frac{8}{d^2-4}} \|\nabla v\|_{S^0(I)}^{\frac{4d}{d^2-4}}.
\end{aligned}$$

This settles (3.52) and hence (3.51). □

We have now all the tools we need to attack Theorem 3.8. As in the mass-critical setting, the stability result for the energy-critical NLS will be obtained iteratively from a short-time perturbation result.

Lemma 3.13 (Short-time perturbations)**.** *Let I be a compact time interval and let $\tilde u$ be an approximate solution to* (3.5) *on $I \times \mathbb{R}^d$ in the sense that*

$$i\tilde u_t + \Delta \tilde u = F(\tilde u) + e$$

for some function e. Assume that

$$\|\tilde u\|_{L_t^\infty \dot H_x^1(I\times\mathbb{R}^d)} \le E$$

for some positive constant E. Moreover, let $t_0 \in I$ and let $u(t_0)$ obey

$$\|u_0 - \tilde u_0\|_{\dot H_x^1} \le E'$$

for some positive constant E'. Assume also the smallness conditions

$$\|\tilde u\|_{X(I)} \le \delta \tag{3.53}$$

$$\big\| e^{i(t-t_0)\Delta}\big(u(t_0) - \tilde u(t_0)\big) \big\|_{X(I)} \le \varepsilon \tag{3.54}$$

$$\|\nabla e\|_{N^0(I)} \le \varepsilon \tag{3.55}$$

for some small $0 < \delta = \delta(E)$ *and* $0 < \varepsilon < \varepsilon_0(E, E')$. *Then there exists a unique solution* $u : I \times \mathbb{R}^d \to \mathbb{C}$ *to* (3.5) *with initial data* $u(t_0)$ *at time* $t = t_0$ *satisfying*

$$\|u - \tilde{u}\|_{X(I)} \lesssim \varepsilon \tag{3.56}$$

$$\|\nabla(u - \tilde{u})\|_{S^0(I)} \lesssim E' \tag{3.57}$$

$$\|\nabla u\|_{S^0(I)} \lesssim E + E' \tag{3.58}$$

$$\|F(u) - F(\tilde{u})\|_{Y(I)} \lesssim \varepsilon \tag{3.59}$$

$$\big\|\nabla\big(F(u) - F(\tilde{u})\big)\big\|_{N^0(I)} \lesssim E'. \tag{3.60}$$

Proof. We will prove the lemma under the additional assumption that $M(u) < \infty$, so that we can rely on Theorem 3.4 to guarantee that u exists. This additional assumption can be removed *a posteriori* by the usual limiting argument: approximate $u(t_0)$ in $\dot{H}^1_x$ by $\{u_n(t_0)\}_n \subseteq H^1_x$ and apply the lemma with $\tilde{u} = u_m$, $u = u_n$, and $e = 0$ to deduce that the sequence of solutions $\{u_n\}_n$ with initial data $\{u_n(t_0)\}_n$ is Cauchy in energy-critical norms and thus convergent to a solution u with initial data $u(t_0)$ which obeys $\nabla u \in S^0(I)$. Thus, it suffices to prove (3.56) through (3.60) as *a priori* estimates, that is we assume that the solution u exists and obeys $\nabla u \in S^0(I)$.

We start by deriving some bounds on $\tilde{u}$ and u. By Strichartz, Lemma 3.11, (3.53), and (3.55),

$$\begin{aligned}
\|\nabla\tilde{u}\|_{S^0(I)} &\lesssim \|\tilde{u}\|_{L^\infty_t \dot{H}^1_x(I\times\mathbb{R}^d)} + \|\nabla F(\tilde{u})\|_{N^0(I)} + \|\nabla e\|_{N^0(I)} \\
&\lesssim E + \|\tilde{u}\|^{\frac{4}{d-2}}_{L^{\frac{2(d+2)}{d-2}}_{t,x}(I\times\mathbb{R}^d)} \|\nabla\tilde{u}\|_{S^0(I)} + \varepsilon \\
&\lesssim E + \delta^{\frac{4c}{d-2}} \|\nabla\tilde{u}\|^{1+\frac{4(1-c)}{d-2}}_{S^0(I)} + \varepsilon,
\end{aligned}$$

where $c = c(d)$ is as in Lemma 3.11. Choosing δ small depending on d, E and ε_0 sufficiently small depending on E, we obtain

$$\|\nabla\tilde{u}\|_{S^0(I)} \lesssim E. \tag{3.61}$$

Moreover, by Lemma 3.10, Lemma 3.12, (3.53), and (3.55),

$$\big\|e^{i(t-t_0)\Delta}\tilde{u}(t_0)\big\|_{X(I)} \lesssim \|\tilde{u}\|_{X(I)} + \|F(\tilde{u})\|_{Y(I)} + \|\nabla e\|_{N^0(I)} \lesssim \delta + \delta^{\frac{d+2}{d-2}} + \varepsilon \lesssim \delta,$$

provided δ and ε_0 are chosen sufficiently small. Combining this with (3.54) and the triangle inequality, we obtain

$$\big\|e^{i(t-t_0)\Delta}u(t_0)\big\|_{X(I)} \lesssim \delta.$$

Thus, another application of Lemma 3.10 combined with Lemma 3.12 gives

$$\|u\|_{X(I)} \lesssim \big\|e^{i(t-t_0)\Delta}u(t_0)\big\|_{X(I)} + \|F(u)\|_{Y(I)} \lesssim \delta + \|u\|^{\frac{d+2}{d-2}}_{X(I)}.$$

Choosing δ sufficiently small, the usual bootstrap argument yields

$$\|u\|_{X(I)} \lesssim \delta. \tag{3.62}$$

Next we derive the claimed bounds on $w := u - \tilde{u}$. Note that w is a solution to

$$\begin{cases} iw_t + \Delta w = F(\tilde{u} + w) - F(\tilde{u}) - e \\ w(t_0) = u(t_0) - \tilde{u}(t_0). \end{cases}$$

Using Lemma 3.10 together with Lemma 3.11 and (3.55), we see that

$$\begin{aligned}\|w\|_{X(I)} &\lesssim \big\|e^{i(t-t_0)\Delta}\big(u(t_0)-\tilde u(t_0)\big)\big\|_{X(I)} + \|\nabla e\|_{N^0(I)} + \|F(u)-F(\tilde u)\|_{Y(I)}\\ &\lesssim \varepsilon + \|F(u)-F(\tilde u)\|_{Y(I)}.\end{aligned}$$

To estimate the difference of the nonlinearities, we use Lemma 3.12, (3.53), and (3.61):

$$\begin{aligned}\|F(u)-F(\tilde u)\|_{Y(I)} &\lesssim \Big[\|\tilde u\|_{X(I)}^{\frac{8}{d^2-4}}\|\nabla\tilde u\|_{S^0(I)}^{\frac{4d}{d^2-4}} + \|w\|_{X(I)}^{\frac{8}{d^2-4}}\|\nabla w\|_{S^0(I)}^{\frac{4d}{d^2-4}}\Big]\|w\|_{X(I)}\\ &\lesssim \delta^{\frac{8}{d^2-4}}E^{\frac{4d}{d^2-4}}\|w\|_{X(I)} + \|\nabla w\|_{S^0(I)}^{\frac{4d}{d^2-4}}\|w\|_{X(I)}^{1+\frac{8}{d^2-4}}. \qquad (3.63)\end{aligned}$$

Thus, choosing δ sufficiently small depending only on E, we obtain

$$\|w\|_{X(I)} \lesssim \varepsilon + \|\nabla w\|_{S^0(I)}^{\frac{4d}{d^2-4}}\|w\|_{X(I)}^{1+\frac{8}{d^2-4}}. \qquad (3.64)$$

On the other hand, by the Strichartz inequality and the hypotheses,

$$\begin{aligned}\|\nabla w\|_{S^0(I)} &\lesssim \|u_0-\tilde u_0\|_{\dot H^1_x} + \|\nabla e\|_{N^0(I)} + \big\|\nabla\big(F(u)-F(\tilde u)\big)\big\|_{N^0(I)}\\ &\lesssim E' + \varepsilon + \big\|\nabla\big(F(u)-F(\tilde u)\big)\big\|_{N^0(I)}. \qquad (3.65)\end{aligned}$$

To estimate the difference of the nonlinearities, we consider low and high dimensions separately. Consider first $3 \le d \le 5$. Using Hölder's inequality followed by Lemma 3.11, (3.53), (3.61), and (3.62),

$$\begin{aligned}&\big\|\nabla\big(F(u)-F(\tilde u)\big)\big\|_{N^0(I)}\\ &\quad\lesssim \big\|\nabla\big(F(u)-F(\tilde u)\big)\big\|_{L_t^{\frac{2d(d+2)}{d^2+2d+4}}L_x^{\frac{2d^2(d+2)}{d^3+4d^2+4d-8}}(I\times\mathbb{R}^d)}\\ &\quad\lesssim \|\nabla\tilde u\|_{S^0(I)}\big(\|u\|_{X^0(I)}+\|\tilde u\|_{X^0(I)}\big)^{\frac{6-d}{d-2}}\|w\|_{X^0(I)} + \|u\|_{X^0(I)}^{\frac{4}{d-2}}\|\nabla w\|_{S^0(I)}\\ &\quad\lesssim \big(E\delta^{\frac{6-d}{d-2}} + \delta^{\frac{4}{d-2}}\big)\|\nabla w\|_{S^0(I)}. \qquad (3.66)\end{aligned}$$

Thus, choosing δ small depending only on E, (3.65) implies

$$\|\nabla w\|_{S^0(I)} \lesssim E' + \varepsilon$$

for $3 \le d \le 5$. Consider now higher dimensions, that is, $d \ge 6$. Using Hölder's inequality followed by Lemma 3.11, (3.61), and (3.62),

$$\begin{aligned}\big\|\nabla\big(F(u)-F(\tilde u)\big)\big\|_{N^0(I)} &\lesssim \big\|\nabla\big(F(u)-F(\tilde u)\big)\big\|_{L_t^{\frac{2d(d+2)}{d^2+2d+4}}L_x^{\frac{2d^2(d+2)}{d^3+4d^2+4d-8}}(I\times\mathbb{R}^d)}\\ &\lesssim \|\nabla\tilde u\|_{S^0(I)}\|w\|_{X^0(I)}^{\frac{4}{d-2}} + \|u\|_{X^0(I)}^{\frac{4}{d-2}}\|\nabla w\|_{S^0(I)}\\ &\lesssim E\|w\|_{X(I)}^{\frac{4}{d-2}} + \delta^{\frac{4}{d-2}}\|\nabla w\|_{S^0(I)}. \qquad (3.67)\end{aligned}$$

Therefore, taking δ sufficiently small, (3.65) implies

$$\|\nabla w\|_{S^0(I)} \lesssim E' + \varepsilon + E\|w\|_{X(I)}^{\frac{4}{d-2}}$$

for $d \ge 6$. Collecting the estimates for low and high dimensions (and choosing $\varepsilon_0 = \varepsilon_0(E')$ sufficiently small), we obtain

$$\|\nabla w\|_{S^0(I)} \lesssim E' + E\|w\|_{X(I)}^{\frac{4}{d-2}} \qquad (3.68)$$

for all $d \ge 3$.

Combining (3.64) with (3.68), the usual bootstrap argument yields (3.56) and (3.57), provided ε_0 is chosen sufficiently small depending on E and E'. By the triangle inequality, (3.57) and (3.61) imply (3.58).

Claims (3.59) and (3.60) follow from (3.63), (3.66), and (3.67) combined with (3.56) and (3.57), provided we take δ sufficiently small depending on E and ε_0 sufficiently small depending on E, E'. □

We are finally in a position to prove the energy-critical stability result.

PROOF OF THEOREM 3.8. Our first goal is to show

$$\|\nabla \tilde{u}\|_{S^0(I)} \leq C(E, L). \tag{3.69}$$

Indeed, by (3.39) we may divide I into $J_0 = J_0(L, \eta)$ subintervals $I_j = [t_j, t_{j+1}]$ such that on each spacetime slab $I_j \times \mathbb{R}^d$

$$\|\tilde{u}\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I_j \times \mathbb{R}^d)} \leq \eta$$

for a small constant $\eta > 0$ to be chosen in a moment. By the Strichartz inequality combined with (3.38) and (3.42),

$$\begin{aligned}
\|\nabla \tilde{u}\|_{S^0(I_j)} &\lesssim \|\tilde{u}(t_j)\|_{\dot{H}^1_x} + \|\nabla e\|_{N^0(I_j)} + \|\nabla F(\tilde{u})\|_{N^0(I_j)} \\
&\lesssim E + \varepsilon + \|\tilde{u}\|^{\frac{4}{d-2}}_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I_j \times \mathbb{R}^d)} \|\nabla \tilde{u}\|_{S^0(I_j)} \\
&\lesssim E + \varepsilon + \eta^{\frac{4}{d-2}} \|\nabla \tilde{u}\|_{S^0(I_j)}.
\end{aligned}$$

Thus, choosing $\eta > 0$ small depending on the dimension d and ε_1 sufficiently small depending on E, we obtain

$$\|\nabla \tilde{u}\|_{S^0(I_j)} \lesssim E.$$

Summing this over all subintervals I_j, we derive (3.69).

Using Lemma 3.11 together with (3.69) and then with (3.40) and (3.41), we obtain

$$\|\tilde{u}\|_{X(I)} \leq C(E, L) \tag{3.70}$$

$$\big\|e^{i(t-t_0)\Delta}\big(u(t_0) - \tilde{u}(t_0)\big)\big\|_{X(I)} \lesssim \varepsilon^{\frac{1}{d+2}} (E')^{\frac{d+1}{d+2}}. \tag{3.71}$$

By (3.70), we may divide I into $J_1 = J_1(E, L)$ subintervals $I_j = [t_j, t_{j+1}]$ such that on each spacetime slab $I_j \times \mathbb{R}^d$

$$\|\tilde{u}\|_{X(I_j)} \leq \delta$$

for some small $\delta = \delta(E) > 0$ as in Lemma 3.13. Moreover, taking $\varepsilon_1(E, E', L)$ sufficiently small compared to $\varepsilon_0(E, C(J_1)E')$, (3.71) guarantees (3.54) with ε replaced by $\varepsilon^c \ll \varepsilon_0$, where c may be taken equal to $\frac{1}{2(d+2)}$. Note that E' is being replaced by $C(J_1)E'$, as the energy of the difference of the two initial data may increase with each iteration.

Thus, choosing ε_1 sufficiently small (depending on J_1, E, and E'), we may apply Lemma 3.13 to obtain for each $0 \le j < J_1$ and all $0 < \varepsilon < \varepsilon_1$,

$$\begin{aligned} \|u - \tilde{u}\|_{X(I_j)} &\le C(j)\varepsilon^c \\ \|u - \tilde{u}\|_{\dot{S}^1(I_j)} &\le C(j)E' \\ \|u\|_{\dot{S}^1(I_j)} &\le C(j)(E + E') \\ \|F(u) - F(\tilde{u})\|_{Y(I_j)} &\le C(j)\varepsilon^c \\ \big\|\nabla\big(F(u) - F(\tilde{u})\big)\big\|_{N^0(I_j)} &\le C(j)E', \end{aligned} \tag{3.72}$$

provided we can show

$$\big\|e^{i(t-t_j)\Delta}\big(u(t_j) - \tilde{u}(t_j)\big)\big\|_{X(I_j)} \lesssim \varepsilon^c \quad \text{and} \quad \|u(t_j) - \tilde{u}(t_j)\|_{\dot{H}^1_x(\mathbb{R}^d)} \lesssim E' \tag{3.73}$$

for each $0 \le j < J_1$. By Lemma 3.10 and the inductive hypothesis,

$$\begin{aligned} &\big\|e^{i(t-t_j)\Delta}\big(u(t_j) - \tilde{u}(t_j)\big)\big\|_{X(I_j)} \\ &\quad\lesssim \big\|e^{i(t-t_0)\Delta}\big(u(t_0) - \tilde{u}(t_0)\big)\big\|_{X(I_j)} + \|\nabla e\|_{N^0(I)} + \|F(u) - F(\tilde{u})\|_{Y([t_0,t_j])} \\ &\quad\lesssim \varepsilon^c + \varepsilon + \sum_{k=0}^{j-1} C(k)\varepsilon^c. \end{aligned}$$

Similarly, by the Strichartz inequality and the inductive hypothesis,

$$\begin{aligned} &\|u(t_j) - \tilde{u}(t_j)\|_{\dot{H}^1_x} \\ &\quad\lesssim \|u(t_0) - \tilde{u}(t_0)\|_{\dot{H}^1_x} + \|\nabla e\|_{N^0([t_0,t_j])} + \big\|\nabla\big(F(u) - F(\tilde{u})\big)\big\|_{N^0([t_0,t_j])} \\ &\quad\lesssim E' + \varepsilon + \sum_{k=0}^{j-1} C(k)E'. \end{aligned}$$

Taking ε_1 sufficiently small depending on J_1, E, and E', we see that (3.73) is satisfied.

Summing the bounds in (3.72) over all subintervals I_j and using Lemma 3.11, we derive (3.43) through (3.45). This completes the proof of the theorem. □

4. A word from our sponsor: Harmonic Analysis

Without doubt, recent progress on nonlinear Schrödinger equations at critical regularity has been made possible by the introduction of important ideas from harmonic analysis, particularly some related to the restriction conjecture.

4.1. The Gagliardo–Nirenberg inequality. The sharp constant for the Gagliardo–Nirenberg inequality was derived by Nagy [**63**], in the one-dimensional setting, and by Weinstein [**105**] for higher dimensions. We begin by recounting this theorem. After that, we will present two applications to nonlinear Schrödinger equations.

Theorem 4.1 (Sharp Gagliardo–Nirenberg, [**63, 105**])**.** *Fix $d \ge 1$ and $0 < p < \infty$ for $d = 1, 2$ or $0 < p < \frac{4}{d-2}$ for $d \ge 3$. Then for all $f \in H^1_x(\mathbb{R}^d)$,*

$$\|f\|_{L^{p+2}_x}^{p+2} \le \tfrac{2(p+2)}{4-p(d-2)}\big(\tfrac{pd}{4-p(d-2)}\big)^{-\frac{pd}{4}} \|Q\|_{L^2_x}^{-p} \|f\|_{L^2_x}^{p+2-\frac{pd}{2}} \|\nabla f\|_{L^2_x}^{\frac{pd}{2}}. \tag{4.1}$$

Here Q denotes the unique positive radial Schwartz solution to $\Delta Q + Q^{p+1} = Q$. Moreover, equality holds in (4.1) if and only if $f(x) = \alpha Q(\lambda(x - x_0))$ for some $\alpha \in \mathbb{C}$, $\lambda \in (0,\infty)$, and $x_0 \in \mathbb{R}^d$.

PROOF. The traditional (non-sharp) Gagliardo–Nirenberg inequality says

$$J(f) := \frac{\|f\|_{L_x^{p+2}}^{p+2}}{\|f\|_{L_x^2}^{p+2-\frac{pd}{2}} \|\nabla f\|_{L_x^2}^{\frac{pd}{2}}} \leq C. \tag{4.2}$$

What we seek here is the optimal constant $C = C_d$ in this inequality. We will present only the proof for $d \geq 2$, following [**105**].

It suffices to consider merely non-negative spherically symmetric functions, since we may replace f by its spherically symmetric decreasing rearrangement f^* (cf. [**54**, §7.17]). The $\dot H_x^1$ norm of f^* is no larger than that of f, while the L_x^2 and L_x^{2+p} norms are invariant under $f \mapsto f^*$. Thus $J(f) \leq J(f^*)$.

Let f_n be an optimizing sequence (of non-negative spherically symmetric functions). By rescaling space and the values of the function, we may assume that $\|\nabla f_n\|_2 = \|f_n\|_2 = 1$. We are now ready for the key step in the argument: The embedding $H^1_{\text{rad}} \hookrightarrow L_x^{2+p}$ is compact; see Lemma A.4. Thus we may deduce that, up to a subsequence, f_n converge strongly in L_x^{2+p}. Additionally, since f_n is an optimizing sequence, we can upgrade the weak convergence of f_n in H_x^1 (courtesy of Alaoglu's theorem) to strong convergence.

In the previous paragraph, we deduced that optimizers exist, that is, there are functions f maximizing $J(f)$. Moreover, f has been normalized to obey $\|\nabla f\|_2 = \|f\|_2 = 1$, which implies $C_d = \|f\|_{p+2}^{p+2}$. By studying small Schwartz-space perturbations of f, we quickly see that any optimizer f must be a distributional solution to

$$(p+2)f^{1+p} - C_d\big\{(p+2-\tfrac{pd}{2})f - \tfrac{pd}{2}\Delta f\big\} = 0. \tag{4.3}$$

This equation can be reduced to $\Delta Q + Q^{p+1} = Q$ by setting

$$f(x) = \alpha^{\frac{1}{p}} Q(\beta^{\frac{1}{2}} x) \quad \text{with } \beta = \tfrac{4-p(d-2)}{pd} \text{ and } \alpha = \tfrac{pd\beta}{2(p+2)} C_d.$$

Taking advantage of $\|f\|_2 = 1$, we may deduce $C_d = \frac{2(p+2)}{4-p(d-2)} \beta^{pd/4} \|Q\|_2^{-p}$.

We now turn to the uniqueness question. It is very tempting to believe that $J(f) \leq J(f^*)$ with equality if and only if $f(x) = e^{i\theta} f^*(x + x_0)$ for some $\theta \in [0, 2\pi)$ and $x_0 \in \mathbb{R}^d$. (This would immediately imply that any optimizer is radially symmetric up to translations.) Alas, it is not true without an additional constraint, for instance, that ∇f^* does not vanish on a set of positive measure; see [**11**]. Fortunately for us, as f^* is a non-zero spherically symmetric solution to (4.3), ∇f^* cannot vanish on a set of positive measure; indeed this is a basic uniqueness property of ODEs.

This leaves us to show uniqueness of positive spherically symmetric solutions of $\Delta Q + Q^{p+1} = Q$, for which we refer the reader to [**49**]. □

Remark. That rearrangement of a non-spherically-symmetric function may fail to reduce the $\dot H_x^1$ norm can be demonstrated with a simple example, which we will now describe. Let $\phi \in C^\infty(\mathbb{R}^d)$ be supported on $\{|x| \leq 2\}$ and obey $\phi(x) = 1$ when $|x| \leq 1$. The skewed 'wedding cake' $f(x) = \phi(x) + \phi(4(x - x_0))$ with $|x_0| \leq \frac{1}{2}$ has $\dot H_x^1$ norm equal to that of its spherically-symmetric decreasing rearrangement.

The main application of Theorem 4.1 in these notes is embodied by the following

Corollary 4.2 (Kinetic energy trapping). *Let $f \in H^1_x(\mathbb{R}^d)$ obey $\|f\|_2 < \|Q\|_2$. Then $\|\nabla f\|_2^2 \lesssim E(f)$, where E denotes the energy associated to the mass-critical focusing NLS. The implicit constant depends only on $\|f\|_2/\|Q\|_2$.*

PROOF. Exercise. □

Combining this with the standard local well-posedness result for subcritical equations and the conservation of mass and energy, we obtain:

Corollary 4.3 (Focusing mass-critical NLS in H^1_x, [**105**]). *For initial data $u(0) \in H^1_x$ obeying $\|u(0)\|_2 < \|Q\|_2$, the focusing mass-critical NLS is globally wellposed.*

PROOF. Exercise. □

Note that this result does not claim that these global solutions scatter. Indeed, scaling shows that scattering for H^1_x initial data is essentially equivalent to scattering for general L^2_x initial data.

4.2. Refined Sobolev embedding. In this subsection, we will describe several refinements of the classical Sobolev embedding inequality. The first is the determination of the optimal constant in that inequality. The following theorem is a special case of results of Aubin [**2**] and Talenti [**86**] (see also [**5, 73**]):

Theorem 4.4 (Sharp Sobolev embedding). *For $d \geq 3$ and $f \in \dot H^1_x(\mathbb{R}^d)$,*

$$\|f\|_{L_x^{\frac{2d}{d-2}}} \leq C_d \|\nabla f\|_{L^2_x} \tag{4.4}$$

with equality if and only if $f = \alpha W(\lambda(x - x_0))$ for some $\alpha \in \mathbb{C}$, $\lambda \in (0, \infty)$, and $x_0 \in \mathbb{R}^d$. Here W denotes

$$W(x) := \big(1 + \tfrac{1}{d(d-2)}|x|^2\big)^{-\frac{d-2}{2}}, \tag{4.5}$$

which is the unique non-negative radial $\dot H^1_x$ solution to $\Delta W + W^{\frac{d+2}{d-2}} = 0$, up to scaling.

In this context, the analogue of Corollary 4.2 is

Corollary 4.5 (Energy trapping, [**38**]). *Assume $E(u_0) \leq (1-\delta_0)E(W)$ for some $\delta_0 > 0$. Then there exists a positive constant δ_1 so that if $\|\nabla u_0\|_2 \leq \|\nabla W\|_2$, then*

$$\|\nabla u_0\|_2^2 \leq (1-\delta_1)\|\nabla W\|_2^2.$$

Here E denotes the energy functional associated to the focusing energy-critical NLS.

PROOF. Exercise. □

We will discuss the proof of Theorem 4.4 in some detail as it is our first brush with our sworn enemy: scaling invariance. First let us note that the argument used to prove Theorem 4.1 will not work here. For instance, $f_n(x) = n^{(d-2)/2} W(nx)$ is a radial optimizing sequence that does not converge. To put it another way, Lemma A.4 fails for $p = \frac{2d}{d-2}$ because of scaling.

There are several proofs of Theorem 4.4. The textbook [**54**] gives an elegant treatment relying on the connection to the Hardy–Littlewood–Sobolev inequality and a (hidden) conformal symmetry. We will be giving a proof that does not rely heavily on rearrangement ideas, since we wish to introduce some techniques that will be important when we discuss improvements to Strichartz inequality.

Lions gave a rearrangement-free proof of the existence of optimizers as one of the first applications of the concentration compactness principle; see [**56**]. The proof we present is a descendant of the one given there. The philosophy underlying concentration compactness has also led to a second kind of refinement to the classical Sobolev embedding, which has proved valuable in the treatment of the energy-critical NLS. The goal is not to understand the maximal possible value of the ratio $J(f) := \|f\|_{2d/(d-2)} \div \|\nabla f\|_2$, but rather for what kind of functions this is big (or equivalently, for which f it is small). Before giving a precise statement, we quickly introduce some of the ideas that will motivate the formulation. We will then revisit the Gagliardo–Nirenberg inequality from this perspective.

Let $A : X \to Y$ be a linear transformation between two Banach spaces. Recall that A is called compact if for every bounded sequence $f_n \in X$, the sequence Af_n has a convergent subsequence. A slightly more convoluted way of saying this is the following.

Exercise. Suppose X is reflexive. Then $A : X \to Y$ is compact if and only if for any bounded sequence $\{f_n\} \subseteq X$ there exists $\phi \in X$ so that along some subsequence $f_n = \phi + r_n$ with $Ar_n \to 0$ in Y. (This may fail if X is not reflexive.)

Even for $2 < q < \frac{2d}{d-2}$, the embedding $H^1_x \hookrightarrow L^q_x$ is not compact since given any non-zero $f \in H^1_x(\mathbb{R}^d)$, the sequence of translates $f_n(x) = f(x - x_n)$, associated to a sequence $x_n \to \infty$ in $\mathbb{R}^d$, is uniformly bounded in $H^1_x(\mathbb{R}^d)$, but has no L^q_x-convergent subsequence. A first attempt to address this failure of compactness, might be to seek a convergent subsequence from among the translates of the original sequence. This does not quite work as can be seen by considering $f_n(x) = \phi_1(x) + \phi_2(x - x_n)$ for some fixed $\phi_1, \phi_2 \in H^1_x(\mathbb{R}^d)$.

Having just seen the example of a sequence that breaks into two 'bubbles' we may begin to despair that a sequence f_n may break into infinitely many small bubbles dancing around $\mathbb{R}^d$ more or less at random. It is time for some good news: $q > 2$, which is to say that in the inequality

$$\|f\|_{L^q_x} \lesssim \|f\|_{L^2_x}^{1-\theta} \|\nabla f\|_{L^2_x}^{\theta}, \quad \theta = \tfrac{(q-2)d}{2q},$$

the power of f integrated on the left-hand side is larger than the power of f and ∇f that is integrated on the right-hand side. The significance of this is that the ℓ^q norm of many small numbers is much much smaller than the ℓ^2 norm of the same collection of numbers. Therefore, a large collection of tiny bubbles whose total H^1_x norm is of order one will have a negligible L^q_x norm.

Theorem 4.6 (The Gagliardo–Nirenberg inequality: bubble decomposition, [**33**])**.** *Fix $d \geq 2$, $2 < q < \frac{2d}{d-2}$, and let f_n be a bounded sequence in $H^1_x(\mathbb{R}^d)$. Then there exist $J^* \in \{0, 1, 2, \ldots\} \cup \{\infty\}$, $\{\phi^j\}_{j=1}^{J^*} \subseteq H^1_x$, and $\{x_n^j\}_{j=1}^{J^*} \subseteq \mathbb{R}^d$ so that along some subsequence in n we may write*

$$f_n(x) = \sum_{j=1}^{J} \phi^j(x - x_n^j) + r_n^J(x) \quad \textit{for all } 0 \leq J \leq J^*, \tag{4.6}$$

where

$$\limsup_{J \to J^*} \limsup_{n \to \infty} \|r_n^J\|_{L^q_x} = 0 \tag{4.7}$$

$$\sup_J \limsup_{n\to\infty} \Big| \|f_n\|_{H^1_x}^2 - \Big(\sum_{j=1}^J \|\phi^j\|_{H^1_x}^2 + \|r_n^J\|_{H^1_x}^2 \Big) \Big| = 0 \tag{4.8}$$

$$\limsup_{J\to J^*} \Big| \limsup_{n\to\infty} \|f_n\|_{L^q_x}^q - \sum_{j=1}^J \|\phi^j\|_{L^q_x}^q \Big| = 0. \tag{4.9}$$

Moreover, for each $j \neq j'$, we have $|x_n^j - x_n^{j'}| \to \infty$. When J^* is finite, we define $\limsup_{J\to J^*} a(J) := a(J^*)$ for any $a : \{0, 1, \dots, J^*\} \to \mathbb{R}$.

We will not make use of this result and we leave its proof to the avid reader who wishes to cement their understanding of the methods described in this subsection. Note that ϕ^j represent the bubbles into which the subsequence is decomposing and J^* is their number. They may be regarded as ordered by decreasing H^1_x norm. The functions r_n^J represent a remainder term, which is guaranteed to be asymptotically irrelevant in L^q_x, but need not converge to zero in H^1_x. This is why r_n^J needs to appear in (4.8), even as $J \to \infty$. Indeed, this is the essence of compactness. Regarding (4.8), we also wish to point out that the divergence of the x_n^j from one another implies that the H^1_x norms of the individual bubbles decouple. That they also decouple from r_n^J is a more subtle statement. It is an expression of the fact that for each pair $j \leq J$,

$$r_n^J(x + x_n^j) \rightharpoonup 0 \quad \text{weakly in } H^1_x,$$

which is built into the way ϕ^j are chosen. (It can also be derived *a posteriori* from the conclusions of this theorem, cf. [**44**, Lemma 2.10].)

The analogue of Theorem 4.6 for Sobolev embedding reads very similarly; it is merely necessary to incorporate the scaling symmetry.

Theorem 4.7 (Sobolev embedding: bubble decomposition, [**26**])**.** *Fix $d \geq 3$ and let f_n be a bounded sequence in $\dot H^1_x(\mathbb{R}^d)$. Then there exist $J^* \in \{0, 1, 2, \dots\} \cup \{\infty\}$, $\{\phi^j\}_{j=1}^{J^*} \subseteq \dot H^1_x$, $\{x_n^j\}_{j=1}^{J^*} \subseteq \mathbb{R}^d$, and $\{\lambda_n^j\}_{j=1}^{J^*} \subseteq (0,\infty)$ so that along some subsequence in n we may write*

$$f_n(x) = \sum_{j=1}^J (\lambda_n^j)^{\frac{2-d}{2}} \phi^j\big((x - x_n^j)/\lambda_n^j\big) + r_n^J(x) \quad \text{for all } 0 \leq J \leq J^* \tag{4.10}$$

with the following five properties:

$$\limsup_{J\to J^*} \limsup_{n\to\infty} \|r_n^J\|_{L_x^{\frac{2d}{d-2}}} = 0 \tag{4.11}$$

$$\sup_J \limsup_{n\to\infty} \Big| \|f_n\|_{\dot H^1_x}^2 - \Big(\|r_n^J\|_{\dot H^1_x}^2 + \sum_{j=1}^J \|\phi^j\|_{\dot H^1_x}^2 \Big) \Big| = 0 \tag{4.12}$$

$$\limsup_{J\to J^*} \limsup_{n\to\infty} \Big| \|f_n\|_{L_x^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} - \sum_{j=1}^J \|\phi^j\|_{L_x^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} \Big| = 0 \tag{4.13}$$

$$\liminf_{n\to\infty} \Big[\frac{|x_n^j - x_n^{j'}|^2}{\lambda_n^j \lambda_n^{j'}} + \frac{\lambda_n^j}{\lambda_n^{j'}} + \frac{\lambda_n^{j'}}{\lambda_n^j} \Big] = \infty \quad \text{for all } j \neq j' \tag{4.14}$$

$$(\lambda_n^j)^{\frac{d-2}{2}} r_n^J\big(\lambda_n^j x + x_n^j\big) \rightharpoonup 0 \quad \text{weakly in } \dot H^1_x \text{ for each } j \leq J. \tag{4.15}$$

Notice that (4.14) says that each pair of bubbles are either widely separated in space or live at very different length scales (or possibly both). This time, we have incorporated the strong form of r_n^J decoupling, (4.15), into the statement of the theorem.

Before embarking on the proofs of Theorems 4.4 and 4.7, let us briefly depart on a small historical excursion. We will, at least, explain why we use the word 'bubble'. In [**76**], Sacks and Uhlenbeck proved the existence of minimal-area spheres in Riemannian manifolds in certain (higher) homotopy classes. They also gave a vivid explanation of why the result is merely for *some* homotopy classes: sometimes the minimal sphere is not really a sphere, but two (or more) spheres joined by one-dimensional geodesic 'umbilical cords'. This obstruction necessitated an ingenious snipping procedure, which can be viewed as an early precursor to the bubble decomposition above. (In this setting, the group of translations is replaced by the group of conformal maps of S^2, that is, of Möbius transformations.)

Minimal surfaces correspond to zero mean curvature. In general, soap films produce surfaces with constant mean curvature. In fact, the mean curvature is proportional to the pressure difference between the two sides; this can be non-zero, as in the case of a spherical bubble. Around the same time as the work of Sacks and Uhlenbeck described above, Wente, [**106**], considered the problem of a large bubble blown on a (comparatively) small wire. He shows that the resulting bubble is asymptotically spherical. The result relies on the extremal property by which the bubble is constructed and, thanks to a subadditivity-type argument deep within the proof, avoids the possibility of multiple bubbles. Consideration of more general (non-extremal) surfaces of constant mean curvature necessitates a full bubble decomposition. This was worked out independently by Brézis and Coron, [**9**], and Struwe, [**85**].

Shortly prior to its appearance in the highly nonlinear setting of constant mean curvature surfaces, Struwe proved a bubble decomposition for the energy-critical elliptic problem $\Delta u + |u|^{\frac{4}{d-2}}u = 0$. This is clearly closely related to Sobolev embedding. Nonetheless, Theorem 4.7 is from [**26**] (building upon some earlier work) as noted above.

As we will see, there is a simple trick for finding the translation parameters x_n^j appearing in (4.10); it uses little more than Hölder's inequality. To deal with the scaling symmetry we need something a little more sophisticated. Littlewood–Paley theory is the natural choice; separating scales is exactly what it does!

Proposition 4.8 (An embedding)**.** *For $d \geq 3$ and $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$\big\|f\big\|_{L_x^{\frac{2d}{d-2}}} \lesssim \big\|\nabla f\big\|_{L_x^2}^{\frac{d-2}{d}} \cdot \sup_{N\in 2^{\mathbb{Z}}} \big\|f_N\big\|_{L_x^{\frac{2d}{d-2}}}^{\frac{2}{d}}. \tag{4.16}$$

Proof. First we give the proof for $d \geq 4$. The key ingredient is the well-known estimate for the Littlewood–Paley square function, Lemma A.7, which we use in the first step. We also use Bernstein's inequality, Lemma A.6.

$$\begin{aligned}
\|f\|_{L_x^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} &\lesssim \int_{\mathbb{R}^d} \Big(\sum_M |f_M|^2\Big)^{\frac{d}{2(d-2)}} \Big(\sum_N |f_N|^2\Big)^{\frac{d}{2(d-2)}}\, dx \\
&\lesssim \sum_{M\leq N} \int_{\mathbb{R}^d} |f_M|^{\frac{d}{d-2}} |f_N|^{\frac{d}{d-2}}\, dx \\
&\lesssim \Big(\sup_{K\in 2^{\mathbb{Z}}} \big\|f_K\big\|_{L_x^{\frac{2d}{d-2}}}\Big)^{\frac{4}{d-2}} \sum_{M\leq N} \big\|f_M\big\|_{L_x^{\frac{2d}{d-4}}} \big\|f_N\big\|_{L_x^2}
\end{aligned}$$

$$\lesssim \Big(\sup_{K\in 2^{\mathbb{Z}}} \big\|f_K\big\|_{L_x^{\frac{2d}{d-2}}}\Big)^{\frac{4}{d-2}} \sum_{M\leq N} M^{-1}N^{-1}\big\|\nabla f_M\big\|_{L_x^{\frac{2d}{d-4}}}\big\|\nabla f_N\big\|_{L_x^2}$$

$$\lesssim \Big(\sup_{K\in 2^{\mathbb{Z}}} \big\|f_K\big\|_{L_x^{\frac{2d}{d-2}}}\Big)^{\frac{4}{d-2}} \sum_{M\leq N} MN^{-1}\big\|\nabla f_M\big\|_{L_x^2}\big\|\nabla f_N\big\|_{L_x^2}$$

$$\lesssim \Big(\sup_{K\in 2^{\mathbb{Z}}} \big\|f_K\big\|_{L_x^{\frac{2d}{d-2}}}\Big)^{\frac{4}{d-2}} \Big(\sum_{K\in 2^{\mathbb{Z}}} \big\|\nabla f_K\big\|_{L_x^2}^2\Big).$$

In passing from the first line to the second, we used that $\frac{d}{2(d-2)} \leq 1$, which is the origin of the restriction $d \geq 4$. To treat three dimensions, one modifies the argument as follows:

$$\begin{aligned}\|f\|_{L_x^6}^6 &\lesssim \int_{\mathbb{R}^d} \Big(\sum_K |f_K|^2\Big)\Big(\sum_M |f_M|^2\Big)\Big(\sum_N |f_N|^2\Big)\,dx \\ &\lesssim \sum_{K\leq M\leq N} \|f_K\|_{L_x^6}\|f_K\|_{L_x^\infty}\|f_M\|_{L_x^6}^2\|f_N\|_{L_x^3}\|f_N\|_{L_x^6} \\ &\lesssim \Big(\sup_{L\in 2^{\mathbb{Z}}} \big\|f_L\big\|_{L_x^6}^4\Big) \sum_{K\leq M\leq N} K^{\frac32}N^{\frac12}\|f_K\|_{L_x^2}\|f_N\|_{L_x^2} \\ &\lesssim \Big(\sup_{L\in 2^{\mathbb{Z}}} \big\|f_L\big\|_{L_x^6}^4\Big) \sum_{K\leq M\leq N} K^{\frac12}N^{-\frac12}\|\nabla f_K\|_{L_x^2}\|\nabla f_N\|_{L_x^2},\end{aligned}$$

which leads to (4.16) via Schur's test and other elementary considerations. □

Our next result introduces the important idea of inverse inequalities. The content of such inequalities is as follows: if a bounded sequence in some strong norm (e.g. $\dot H_x^1$) does not converge weakly to zero in a weaker norm (e.g., $L_x^{2d/(d-2)}$), then this can be attributed to the sequence containing a bubble of concentration. While we have not seen the following precise statement in print, it is a natural off-shoot of existing ideas.

Proposition 4.9 (Inverse Sobolev Embedding)**.** *Fix $d \geq 3$ and let $\{f_n\} \subseteq \dot H_x^1(\mathbb{R}^d)$. If*

$$\lim_{n\to\infty} \|f_n\|_{\dot H_x^1(\mathbb{R}^d)} = A \quad \textit{and} \quad \liminf_{n\to\infty} \|f_n\|_{L_x^{\frac{2d}{d-2}}(\mathbb{R}^d)} = \varepsilon, \tag{4.17}$$

then there exist a subsequence in n, $\phi \in \dot H_x^1(\mathbb{R}^d)$, $\{\lambda_n\} \subseteq (0,\infty)$, and $\{x_n\} \subseteq \mathbb{R}^d$ so that along the subsequence, we have the following three properties:

$$\lambda_n^{\frac{d-2}{2}} f_n(\lambda_n x + x_n) \rightharpoonup \phi(x) \quad \textit{weakly in } \dot H_x^1(\mathbb{R}^d) \tag{4.18}$$

$$\lim_{n\to\infty}\Big[\big\|f_n(x)\big\|_{\dot H_x^1}^2 - \big\|f_n(x) - \lambda_n^{\frac{2-d}{2}}\phi\big(\lambda_n^{-1}(x-x_n)\big)\big\|_{\dot H_x^1}^2\Big] = \|\phi\|_{\dot H_x^1}^2 \gtrsim A^2\big(\tfrac{\varepsilon}{A}\big)^{\frac{d^2}{2}} \tag{4.19}$$

$$\limsup_{n\to\infty}\Big\|f_n(x) - \lambda_n^{\frac{2-d}{2}}\phi\big(\lambda_n^{-1}(x-x_n)\big)\Big\|_{L_x^{\frac{2d}{d-2}}(\mathbb{R}^d)}^{\frac{2d}{d-2}} \leq \varepsilon^{\frac{2d}{d-2}}\Big[1 - c\big(\tfrac{\varepsilon}{A}\big)^{\frac{d(d+2)}{2}}\Big]. \tag{4.20}$$

Here c is a (dimension-dependent) constant.

Proof. By passing to a subsequence, we may assume that $\|f_n\|_{\frac{2d}{d-2}} \to \varepsilon$ from the very beginning. This will not be important until we turn our attention to (4.20).

By Proposition 4.8, there exists $\{N_n\} \subseteq 2^{\mathbb{Z}}$ so that

$$\liminf_{n\to\infty} \|P_{N_n} f_n\|_{L_x^{\frac{2d}{d-2}}(\mathbb{R}^d)} \gtrsim \varepsilon^{\frac{d}{2}} A^{-\frac{d-2}{2}}.$$

We set $\lambda_n = N_n^{-1}$. To find x_n, we use Hölder's inequality:

$$\begin{aligned} \varepsilon^{\frac{d}{2}} A^{-\frac{d-2}{2}} &\lesssim \liminf_{n\to\infty} \big\|P_{N_n} f_n\big\|_{L_x^{\frac{2d}{d-2}}(\mathbb{R}^d)} \\ &\lesssim \liminf_{n\to\infty} \big\|P_{N_n} f_n\big\|_{L_x^2(\mathbb{R}^d)}^{\frac{d-2}{d}} \big\|P_{N_n} f_n\big\|_{L_x^\infty(\mathbb{R}^d)}^{\frac{2}{d}} \\ &\lesssim \liminf_{n\to\infty} \big(AN_n^{-1}\big)^{\frac{d-2}{d}} \big\|P_{N_n} f_n\big\|_{L_x^\infty(\mathbb{R}^d)}^{\frac{2}{d}}. \end{aligned}$$

That is, there exists $x_n \in \mathbb{R}^d$ so that

$$\liminf_{n\to\infty} N_n^{\frac{2-d}{2}} \big|[P_{N_n} f_n](x_n)\big| \gtrsim \varepsilon^{\frac{d^2}{4}} A^{1-\frac{d^2}{4}}. \tag{4.21}$$

Having chosen the parameters λ_n and x_n, Alaoglu's theorem guarantees that (4.18) holds for some subsequence in n and some $\phi \in \dot H_x^1$. To see that ϕ is non-zero, let us write k for the convolution kernel of the Littlewood–Paley projection onto frequencies of size one. That is, let $k := P_1\delta_0$. Using (4.21) we obtain

$$\begin{aligned} |\langle k, \phi\rangle| &= \lim_{n\to\infty} \bigg| \int_{\mathbb{R}^d} \bar k(x) N_n^{-\frac{d-2}{2}} f_n(x_n + N_n^{-1}x)\, dx \bigg| \\ &= \lim_{n\to\infty} N_n^{\frac{2-d}{2}} \bigg| \int_{\mathbb{R}^d} N_n^d \bar k\big(N_n(y-x_n)\big) f_n(y)\, dy \bigg| \\ &= \lim_{n\to\infty} N_n^{\frac{2-d}{2}} \big|[P_{N_n} f_n](x_n)\big| \\ &\gtrsim \varepsilon^{\frac{d^2}{4}} A^{1-\frac{d^2}{4}}. \end{aligned}$$

This implies that $\|\nabla\phi\|_2 \gtrsim \|\phi\|_{\frac{2d}{d-2}} \gtrsim \varepsilon^{\frac{d^2}{4}} A^{1-\frac{d^2}{4}}$. To deduce (4.19) we apply the following basic Hilbert-space fact:

$$g_n \rightharpoonup g \quad \Longrightarrow \quad \|g_n\|^2 - \|g - g_n\|^2 \to \|g\|^2 \tag{4.22}$$

with $g_n := \lambda_n^{\frac{d-2}{2}} f_n(\lambda_n x + x_n)$.

To obtain (4.20), we are going to need to work a little harder (cf. the warning below). First we note that since g_n is bounded in $\dot H_x^1(\mathbb{R}^d)$, we may pass to a further subsequence so that $g_n \to \phi$ in L_x^2-sense on any compact set (via the Rellich–Kondrashov Theorem). By passing to yet another subsequence, we can then guarantee that $g_n \to \phi$ almost everywhere in $\mathbb{R}^d$. Thus we may apply Lemma A.5 to obtain

$$\limsup_{n\to\infty} \Big\| \lambda_n^{\frac{d-2}{2}} f_n(\lambda_n x + x_n) - \phi(x) \Big\|_{L_x^{\frac{2d}{d-2}}(\mathbb{R}^d)}^{\frac{2d}{d-2}} = \varepsilon^{\frac{2d}{d-2}} - \big\|\phi\big\|_{L_x^{\frac{2d}{d-2}}(\mathbb{R}^d)}^{\frac{2d}{d-2}}.$$

This gives (4.20) after taking into account the invariance of the norm under symmetries. □

Warning. It is very tempting to believe that extracting a bubble automatically reduces the $L_x^q(\mathbb{R}^d)$ norm, which is to say that some adequate analogue of (4.22) holds outside of Hilbert spaces. This is not the case; indeed, for $1 \le q < \infty$,

$$\Big(g_n \rightharpoonup g \text{ in } L_x^q \;\; \Rightarrow \;\; \limsup \big[\|g_n\|_{L_x^q} - \|g_n - g\|_{L_x^q}\big] \ge 0 \Big) \Rightarrow q = 2. \tag{4.23}$$

To see this, it suffices to consider the case where g_n and g are supported on the same unit cube and where g is equal to a constant there. Under these restrictions, (4.23) reduces to the following probabilistic statement:

$$\Big(\mathbb{E}\big\{|X|^q\big\} \geq \mathbb{E}\big\{|X - \mathbb{E}(X)|^q\big\} \text{ for all random variables } X\Big) \Rightarrow q = 2.$$

This in turn can be verified by a random variable taking only two values. Indeed, let X be the random variable defined by $X = 2$ with probability p and $X = -1$ with probability $1 - p$ and consider p close to $\frac{1}{3}$.

With Proposition 4.9 in hand, we will be able to quickly complete the

Proof of Theorem 4.7. As $\|\nabla f_n\|_2$ is a bounded sequence, we may pass to a subsequence so that it converges. Applying Proposition 4.9 recursively leads to

$$\begin{aligned} f_n^1 &:= f_n(x) - (\lambda_n^1)^{\frac{2-d}{2}} \phi^1\big((x - x_n^1)/\lambda_n^1\big) \\ f_n^2 &:= f_n^1(x) - (\lambda_n^2)^{\frac{2-d}{2}} \phi^2\big((x - x_n^2)/\lambda_n^2\big) \\ &\;\;\vdots \\ f_n^{j+1} &:= f_n^j(x) - (\lambda_n^j)^{\frac{2-d}{2}} \phi^j\big((x - x_n^j)/\lambda_n^j\big), \end{aligned}$$

where in passing from each iteration to the next we successively require n to lie in an ever smaller (infinite!) subset of the integers. This process terminates (and J^* is finite) as soon as we have $\liminf_{n\to\infty} \|f_n^{j_0}\|_{\frac{2d}{d-2}} = 0$; indeed, $J^* = j_0$. In this case we restrict n to lie in the final subsequence. If instead $J^* = \infty$, we simply restrict n to lie in the diagonal subsequence.

Setting $r_n^0 := f_n$ and $r_n^J := f_n^J$ for $1 \leq J \leq J^*$, it remains to check the various conclusions of the theorem. Equation (4.11) is inherited directly from (4.20). We turn now to (4.14); this is a consequence of (4.18) and the fact that (by our choice of J^*) all ϕ^j are non-zero. Claim (4.15) follows from (4.14) and (4.20). Next, by approximating ϕ^j by C_c^∞ functions, it is not difficult to deduce (4.13) from (4.11) and (4.14). Lastly, (4.12) follows from (4.14) and (4.15) together with (4.22). □

Proof of Theorem 4.4. The key point is to show the existence of optimizers; once this is known, one may repeat the arguments from Theorem 4.1.

Let f_n be a maximizing sequence for the ratio

$$J(f) := \|f\|_{L_x^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} \div \|\nabla f\|_{L_x^2}^{\frac{2d}{d-2}}$$

with $\|\nabla f_n\|_2 \equiv 1$. Applying Theorem 4.7 and passing to the requisite subsequence, we find

$$\sup_f J(f) = \lim_{n\to\infty} J(f_n) = \sum_{j=1}^{\infty} \big\|\phi^j\big\|_{L_x^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} \leq \sup_f J(f) \sum_{j=1}^{\infty} \big\|\nabla\phi^j\big\|_{L_x^2}^{\frac{2d}{d-2}}. \tag{4.24}$$

We also find $\sum_{j=1}^\infty \|\nabla\phi^j\|_2^2 \leq 1$, where the inequality stems from the omission of r_n^J. Combining these two observations with $\frac{2d}{d-2} > 2$, we see that only one of the ϕ^j may have non-zero norm; indeed, we must also have $\|\nabla\phi^j\|_2 = 1$. Thus f_n can be made to converge strongly by applying symmetries to each function. This confirms the existence of an optimizer. □

While Proposition 4.8 seems a little odd, it is well suited to proving Theorem 4.7, as we saw. To finish this subsection, we will describe some more natural improved Sobolev embeddings. These are expressed in terms of Besov norms,

$$\|f\|_{\dot{B}^s_{p,q}} := \Big(\sum_{N\in 2^{\mathbb{Z}}} \big\| N^s f_N \big\|^q_{L^p_x} \Big)^{\frac{1}{q}},$$

though we will not presuppose any familiarity with Besov spaces. The following result is a strengthening of Sobolev embedding in terms of Besov spaces (cf. [**48**, p. 56] or [**99**, p. 170]):

Proposition 4.10 (Besov embedding)**.** *For $d \geq 3$ and $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$\|f\|^{\frac{2d}{d-2}}_{L_x^{\frac{2d}{d-2}}} \lesssim \sum_{N\in 2^{\mathbb{Z}}} \big\| N f_N \big\|^{\frac{2d}{d-2}}_{L^2_x} \sim \sum_{N\in 2^{\mathbb{Z}}} \big\| \nabla f_N \big\|^{\frac{2d}{d-2}}_{L^2_x} \tag{4.25}$$

That is, $\dot{B}^1_{2,2d/(d-2)} \hookrightarrow L_x^{2d/(d-2)}$.

Proof. Exercise: prove this result by mimicking the proof of Proposition 4.8. □

By applying Hölder's inequality to the sum over $2^{\mathbb{Z}}$, we see that this proposition directly implies $\dot{B}^1_{2,q} \hookrightarrow L_x^{2d/(d-2)}$ for any $q \leq \frac{2d}{d-2}$ (e.g., $q = 2$ corresponds to the usual Sobolev embedding). Larger values of q are forbidden, as can be seen by considering a linear combination of many many bumps that are well separated both in space and in characteristic length scale. In this sense, the embedding given above is sharp.

The following variant of Proposition 4.10 forms the basis for the proof of Theorem 4.7 in [**26**]; see [**26**, Proposition 3.1] or [**27**, Théorème 1].

Corollary 4.11 (Interpolated Besov embedding, [**27**])**.** *For $d \geq 3$ and $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$\big\|f\big\|_{L_x^{\frac{2d}{d-2}}} \lesssim \big\|f\big\|^{1-\frac{2}{d}}_{\dot{H}^1_x} \cdot \sup_{N\in 2^{\mathbb{Z}}} \big\|\nabla f_N\big\|^{\frac{2}{d}}_{L^2_x} \sim \big\|f\big\|^{1-\frac{2}{d}}_{\dot{B}^1_{2,2}} \big\|f\big\|^{\frac{2}{d}}_{\dot{B}^1_{2,\infty}}. \tag{4.26}$$

Proof. Exercise ×2: deduce this from Proposition 4.10 and then independently from Proposition 4.8. □

Note that relative to Proposition 4.8, the only difference is that the supremum factor contains the $\dot{H}^1_x$ norm rather than the $L_x^{2d/(d-2)}$ norm. It is this change that allowed us to include (4.20) in Proposition 4.9, which in turn simplified the proof of Theorem 4.7.

4.3. In praise of stationary phase.

Although we are blessed with a simple exact formula for the kernel of the free propagator $e^{it\Delta}$,

$$e^{it\Delta}(x,y) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)-it|\xi|^2}\,d\xi = (4\pi i t)^{-d/2} e^{i|x-y|^2/4t}, \tag{4.27}$$

many of its properties are more clearly visible from the method of stationary phase.

Our first result is perhaps the best known of this genre. The name we use originates in optics, where it describes diffraction patterns in the (monochromatic) paraxial approximation. In particular, it shows how a laser pointer can be used to draw Fourier transforms.

Lemma 4.12 (Fraunhofer formula)**.** *For $\psi \in L^2_x(\mathbb{R}^d)$ and $t \to \pm\infty$,*

$$\big\| [e^{it\Delta}\psi](x) - (2it)^{-\frac{d}{2}} e^{i|x|^2/4t} \hat{\psi}\big(\tfrac{x}{2t}\big) \big\|_{L^2_x} \to 0. \tag{4.28}$$

Proof. While this asymptotic is most easily understood in terms of stationary phase, the simplest proof dodges around this point. By (4.27), we have the identity

$$\begin{aligned} \text{LHS(4.28)} &= \Big\| (4\pi i t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i|x-y|^2/4t} [1 - e^{-i|y|^2/4t}] \psi(y)\, dy \Big\|_{L^2_x} \\ &= \Big\| \int_{\mathbb{R}^d} e^{it\Delta}(x,y)\; [1 - e^{-i|y|^2/4t}] \psi(y)\, dy \Big\|_{L^2_x} \\ &= \big\| [1 - e^{-i|y|^2/4t}] \psi(y) \big\|_{L^2_y}. \end{aligned} \tag{4.29}$$

The result now follows from the dominated convergence theorem. □

The Fraunhofer formula clearly shows that wave packets centered at frequency ξ travel with velocity 2ξ. That is, the *group* velocity is 2ξ, in the usual jargon. By comparison, plane wave solutions, $e^{i\xi\cdot(x-\xi t)}$, travel at the *phase* velocity ξ. As one last piece of jargon, we define the *dispersion relation*: it is the relation $\omega = \omega(\xi)$, so that plane wave solutions take the form $e^{i\xi\cdot x - i\omega t}$. In particular, for the Schrödinger equation, $\omega = |\xi|^2$.

The remaining two results in this subsection are both expressions of the dispersive nature of the free propagator, that is, of the fact that different frequencies travel at different speeds. In the first instance, this is quite clear. The second result shows that high-frequency waves spend little time near the spatial origin.

Lemma 4.13 (Kernel estimates)**.** *For any $m \geq 0$, the kernel of the linear propagator obeys the following estimates:*

$$\big|(P_N e^{it\Delta})(x,y)\big| \lesssim_m \begin{cases} |t|^{-d/2} & : \ |x-y| \sim N|t| \geq N^{-1} \\ \dfrac{N^d}{\langle N^2 t\rangle^m \langle N|x-y|\rangle^m} & : \ \textit{otherwise.} \end{cases} \tag{4.30}$$

Proof. Exercise in stationary phase. □

Proposition 4.14 (Local Smoothing, [**21, 79, 100**])**.** *Fix $\varphi \in C^\infty_c(\mathbb{R}^d)$. Then for all $f \in L^2_x(\mathbb{R}^d)$ and $R > 0$,*

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} \big| \big[|\nabla|^{\frac12} e^{it\Delta} f \big](x) \big|^2 \varphi(x/R)\, dx\, dt \lesssim_\varphi R \|f\|^2_{L^2_x(\mathbb{R}^d)} \tag{4.31}$$

and so,

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} \big| \big[|\nabla|^{\frac12} e^{it\Delta} f \big](x) \big|^2 \langle x\rangle^{-1-\varepsilon}\, dx\, dt \lesssim_\varepsilon \|f\|^2_{L^2_x(\mathbb{R}^d)} \tag{4.32}$$

for any $\varepsilon > 0$.

Proof. Both (4.31) and (4.32) follow from the same argument (though the second can also be deduced from the first by summing over dyadic R): Given $a : \mathbb{R}^d \to [0,\infty)$,

$$\iint \big| \big[|\nabla|^{\frac12} e^{it\Delta} f \big](x) \big|^2 a(x)\, dx\, dt \sim \iint \frac{|\xi|^{\frac12}|\eta|^{\frac12}}{|\xi| + |\eta|} \hat{a}(\eta - \xi) \delta(|\xi| - |\eta|) \hat{f}(\xi) \overline{\hat{f}(\eta)}\, d\xi\, d\eta.$$

The result now follows from Schur's test. □

Exercise. Show that for $d \geq 2$, one may make the replacement $|\nabla| \mapsto \langle \nabla \rangle$ in (4.32) provided one also requires $\varepsilon \geq 1$.

The next result is Lemma 3.7 from [**41**] extended to all dimensions. This will be used in the proof of Lemma 5.7. We give a quantitative proof.

Corollary 4.15. *Given* $\phi \in \dot H^1_x(\mathbb{R}^d)$,

$$\|\nabla e^{it\Delta}\phi\|^3_{L^2_{t,x}([-T,T]\times\{|x|\leq R\})} \lesssim T^{\frac{2}{d+2}} R^{\frac{3d+2}{2(d+2)}} \|e^{it\Delta}\phi\|_{L^{2(d+2)/(d-2)}_{t,x}} \|\nabla\phi\|^2_{L^2_x}.$$

Proof. Given $N > 0$, Hölder's and Bernstein's inequalities imply

$$\begin{aligned}\|\nabla e^{it\Delta}\phi_{<N}\|_{L^2_{t,x}([-T,T]\times\{|x|\leq R\})} &\lesssim T^{2/(d+2)} R^{2d/(d+2)} \|e^{it\Delta}\nabla\phi_{<N}\|_{L^{2(d+2)/(d-2)}_{t,x}} \\ &\lesssim T^{2/(d+2)} R^{2d/(d+2)}\, N\, \|e^{it\Delta}\phi\|_{L^{2(d+2)/(d-2)}_{t,x}}.\end{aligned}$$

On the other hand, the high frequencies can be estimated using local smoothing:

$$\begin{aligned}\|\nabla e^{it\Delta}\phi_{\geq N}\|_{L^2_{t,x}([-T,T]\times\{|x|\leq R\})} &\lesssim R^{1/2}\||\nabla|^{1/2}\phi_{\geq N}\|_{L^2_x} \\ &\lesssim N^{-1/2}R^{1/2}\|\nabla\phi\|_{L^2_x}.\end{aligned}$$

The result now follows by optimizing the choice of N. □

4.4. Improved Strichartz inequalities. Let us begin by recalling the original Strichartz inequality in a slightly different formulation (cf. Theorem 3.2).

Theorem 4.16 (Strichartz)**.** *Fix* $2 \leq q, r, \tilde q, \tilde r \leq \infty$ *with* $\frac{2}{q} + \frac{d}{r} = \frac{2}{\tilde q} + \frac{d}{\tilde r} = \frac{d}{2}$. *If* $d = 2$, *we also require that* $q, \tilde q > 2$. *Then*

$$\big\|e^{it\Delta}u_0\big\|_{L^q_tL^r_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|u_0\|_{L^2_x(\mathbb{R}^d)} \tag{4.33}$$

$$\Big\|\int_{\mathbb{R}} e^{-it\Delta}F(t)\,dt\Big\|_{L^2_x(\mathbb{R}^d)} \lesssim \|F\|_{L^{q'}_tL^{r'}_x(\mathbb{R}\times\mathbb{R}^d)} \tag{4.34}$$

$$\Big\|\int_{s<t} e^{i(t-s)\Delta}F(s)\,ds\Big\|_{L^q_tL^r_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|F\|_{L^{\tilde q'}_tL^{\tilde r'}_x(\mathbb{R}\times\mathbb{R}^d)} \tag{4.35}$$

for all $u_0 \in \mathcal{S}(\mathbb{R}^d)$ *and* $F \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^d)$.

Proof. We treat the case $q, \tilde q > 2$. The endpoint case is more involved; see [**37**].

The linear operators in (4.33) and (4.34) are adjoints of one another; thus, by the method of TT^* both will follow once we prove

$$\Big\|\int_{\mathbb{R}} e^{i(t-s)\Delta}F(s)\,ds\Big\|_{L^q_tL^r_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|F\|_{L^{q'}_tL^{r'}_x(\mathbb{R}\times\mathbb{R}^d)}. \tag{4.36}$$

By the dispersive estimate (3.2) and then the Hardy-Littlewood-Sobolev inequality, we have

$$\text{LHS}(4.36) \lesssim \Big\|\int_{\mathbb{R}} |t-s|^{\frac{d}{r}-\frac{d}{2}}\|F(s)\|_{L^{r'}_x}\,ds\Big\|_{L^q_t(\mathbb{R})} \lesssim \text{RHS}(4.36).$$

The argument just presented also covers (4.35) in the case $q = \tilde q$, $r = \tilde r$. To go beyond this case, it helps to consider the estimate in dualized form:

$$\Big|\iint_{s<t} \langle e^{i(t-s)\Delta}F(s), G(t)\rangle\,ds\,dt\Big| \lesssim \|F\|_{L^{q'}_tL^{r'}_x(\mathbb{R}\times\mathbb{R}^d)}\|G\|_{L^{\tilde q'}_tL^{\tilde r'}_x(\mathbb{R}\times\mathbb{R}^d)}. \tag{4.37}$$

The case $\tilde{q} = \infty$, $\tilde{r} = 2$ follows from (4.34):

$$\text{LHS}(4.37) \leq \Big\| \int_{s<t} e^{i(t-s)\Delta} F(s)\, ds \Big\|_{L_t^\infty L_x^2} \|G\|_{L_t^1 L_x^2} \lesssim \|F\|_{L_t^{q'} L_x^{r'}} \|G\|_{L_t^1 L_x^2}$$

Interpolating between this and the case $q = \tilde{q}$ mentioned above proves (4.35) for all exponents where $q \leq \tilde{q}$. The other case may be deduced symmetrically. □

The main purpose of this subsection is to discuss some variants and extensions of Theorem 4.16. While (4.33) and (4.34) do not hold for any larger class of exponents, (4.35) does. Indeed, this fact plays an important role in the proof of the endpoint case, [**37**]. We have seen one instance of this already, namely, Lemma 3.10. For the largest set of exponents currently known (and a discussion of counterexamples), see [**25, 101**].

One may also consider changing the norm on the right-hand side of (4.33). Placing u_0 in an L_x^p space, brings us back to the dispersive estimate, (3.2). Asking for bounds in terms of $\hat{u}_0$ leads us directly to a profound question:

Conjecture 4.17 (Stein's Restriction Conjecture, [**80**])**.**

$$\|e^{it\Delta} f\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|\hat{f}\|_{L^p_\xi(\mathbb{R}^d)} \tag{4.38}$$

provided $\frac{d+2}{d} p' = q > \frac{2(d+1)}{d}$.

Despite intensive effort, this conjecture remains unresolved except when $d = 1$, [**24, 109**]. To date, the best result we know is that the conjecture holds for $q > \frac{2(d+3)}{d+1}$, [**88**]. The proof of this takes advantage of a certain bilinear estimate, which we reproduce below as Theorem 4.20.

A variety of bilinear estimates have played an important role in the treatment of mass- and energy-critical NLS. The first such estimate we give appears as [**66**, Theorem 2] in the one dimensional setting, as [**6**, Lemma 111] for $d = 2$, and as [**20**, Lemma 3.4] for general dimensions. We postpone further discussion until after Corollary 4.19.

Theorem 4.18 (Bilinear Strichartz I, [**6, 20, 66**])**.** *Fix $d \geq 1$ and $M \leq N$, then*

$$\big\| [e^{it\Delta} P_M f][e^{it\Delta} P_N g] \big\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim M^{\frac{d-1}{2}} N^{-\frac{1}{2}} \|f\|_{L^2_x(\mathbb{R}^d)} \|g\|_{L^2_x(\mathbb{R}^d)} \tag{4.39}$$

When $d = 1$ we require $M \leq \frac{1}{4}N$, so that $P_N P_M = 0$.

Proof. For $M \sim N$ and $d \neq 1$, the result follows from the $L_x^2 \to L_t^4 L_x^{\frac{2d}{d-1}}$ Strichartz inequality and Bernstein.

Turning to the case $M \leq \frac{1}{4}N$, we note that by duality and the Parseval identity, it suffices to show

$$\begin{aligned} \Big| \iint_{\mathbb{R}^d\times\mathbb{R}^d} & F(|\xi|^2 + |\eta|^2, \xi+\eta) \widehat{f_M}(\xi) \widehat{g_N}(\eta)\, d\xi\, d\eta \Big| \\ & \lesssim M^{\frac{d-1}{2}} N^{-\frac{1}{2}} \|F\|_{L^2_{\omega,\xi}(\mathbb{R}^{1+d})} \|\hat{f}\|_{L^2_\xi(\mathbb{R}^d)} \|\hat{g}\|_{L^2_\xi(\mathbb{R}^d)}. \end{aligned} \tag{4.40}$$

Indeed, by breaking the region of integration into several pieces (and rotating the coordinate system appropriately), we may restrict the region of integration to a set where $\eta_1 - \xi_1 \gtrsim N$. Next, we make the change of variables $\zeta = \xi + \eta$, $\omega = |\xi|^2 + |\eta|^2$, and $\beta = (\xi_2, \ldots, \xi_d)$. Note that $|\beta| \lesssim M$ while the Jacobian is $J \sim N^{-1}$. Using this information together with Cauchy–Schwarz:

$$\begin{aligned}
\text{LHS}(4.40) &= \Big| \iiint F(\omega,\zeta)\widehat{f_M}(\xi)\widehat{g_N}(\eta) J \, d\omega \, d\zeta \, d\beta \Big| \\
&\leq \|F\|_{L^2_{\omega,\xi}(\mathbb{R}^{1+d})} \int \Big[\iint |\widehat{f_M}(\xi)|^2 |\widehat{g_N}(\eta)|^2 J^2 \, d\omega \, d\zeta \Big]^{\frac12} d\beta \\
&\lesssim \|F\|_{L^2_{\omega,\xi}(\mathbb{R}^{1+d})} M^{\frac{d-1}{2}} \Big(\iiint |\widehat{f_M}(\xi)|^2 |\widehat{g_N}(\eta)|^2 J^2 \, d\omega \, d\zeta \, d\beta \Big)^{\frac12} \\
&\lesssim \|F\|_{L^2_{\omega,\xi}(\mathbb{R}^{1+d})} M^{\frac{d-1}{2}} \Big(\iint |\widehat{f_M}(\xi)|^2 |\widehat{g_N}(\eta)|^2 N^{-1} \, d\xi \, d\eta \Big)^{\frac12},
\end{aligned}$$

which implies (4.39). □

Corollary 4.19 (Bilinear Strichartz, II)**.** *Let M, N, and d be as above. Given any spacetime slab $I \times \mathbb{R}^d$, any $t_0 \in I$, and any functions u, v defined on $I \times \mathbb{R}^d$,*

$$\begin{aligned}
\|(P_{\geq N} u)(P_{\leq M} v)\|_{L^2_{t,x}} \lesssim M^{\frac{d-1}{2}} N^{-\frac12} \Big(\|P_{\geq N} u(t_0)\|_{L^2_x} + \|(i\partial_t + \Delta) P_{\geq N} u\|_{L^{\frac{2(d+2)}{d+4}}_{t,x}} \Big) \\
\times \Big(\|P_{\leq M} v(t_0)\|_{L^2_x} + \|(i\partial_t + \Delta) P_{\leq M} v\|_{L^{\frac{2(d+2)}{d+4}}_{t,x}} \Big),
\end{aligned}$$

where all spacetime norms are taken over $I \times \mathbb{R}^d$.

Proof. See [**104**, Lemma 2.5], which builds on earlier versions in [**8, 20**]. □

We now embark on a brief discussion of Theorem 4.18. The total power of M and N in (4.39) is dictated by scaling; the point here is that we can skew it heavily in favour of M, thereby obtaining smallness when $M \ll N$. Results of this type have played a vital role in the treatment of mass- and energy-critical NLS, because they have made it possible to 'break' the scaling symmetry. More precisely, Theorem 4.18 shows that interactions between widely separated scales are suppressed, thus, ultimately, permitting one to focus on a single scale at a time. We have already seen a related example of such spontaneous symmetry breaking in the previous subsection (and will see another shortly), namely, that individual optimizers in the Sobolev embedding inequality fail to be dilation/translation invariant; indeed, they have a very definite location and intrinsic length scale.

The particular bilinear estimate given in Theorem 4.18 has proved more useful for the energy-critical NLS than for the mass-critical problem. For the mass-critical NLS, we need a different kind of bilinear estimate:

Theorem 4.20 (Bilinear Restriction, [**88**])**.** *Let $f, g \in L^2_x(\mathbb{R}^d)$. Suppose that for some $c > 0$,*

$$N := \operatorname{dist}(\operatorname{supp} \hat f, \operatorname{supp} \hat g) \geq c \max\{\operatorname{diam}(\operatorname{supp} \hat f), \operatorname{diam}(\operatorname{supp} \hat g)\}.$$

Then for $q > \frac{d+3}{d+1}$,

$$\big\| [e^{it\Delta} f][e^{it\Delta} g] \big\|_{L^q_{t,x}} \lesssim_c N^{d - \frac{d+2}{q}} \|f\|_{L^2_x} \|g\|_{L^2_x}$$

Remarks. 1. For a fuller discussion of this result and its context, see [**88, 93**]. In particular, we note that Theorem 4.20 was conjectured by Klainerman and Machedon and that Tao indicates that his work was inspired by the analogous result for the wave equation, [**107**].

2. For $q = \frac{d+2}{d}$ (or greater) this follows from Theorem 4.16 (and Bernstein). The point here is that some $q < \frac{d+2}{d}$ are allowed.

3. Whether the theorem remains true for $q = \frac{d+3}{d+1}$ is currently open (except when $d = 1$); however it does fail for q smaller (cf. [**93**, §2.7]). The picture to have in mind is of one train overtaking another: two wave packets that are long in the common direction of propagation (though not so large in the transverse direction) travelling at different speeds. More precisely, consider

$$f = \delta^{\frac{d+1}{2}}\phi(\delta^2 x_1)\phi(\delta x_2)\cdots\phi(\delta x_d) \quad \text{and} \quad g = \delta^{\frac{d+1}{2}} e^{ix_1}\phi(\delta^2 x_1)\phi(\delta x_2)\cdots\phi(\delta x_d)$$

with $\hat{\phi} \in C^\infty(\mathbb{R})$ of compact support and $\delta \downarrow 0$. Note that if the wave packets are made more slender in the transverse direction, they will disperse too quickly.

We will not even attempt to outline the proof of Theorem 4.20; however, we will endeavour to provide a reasonable description of how it is used in the treatment of NLS. To do this, we need to introduce the standard family of dyadic cubes, which we do next. After that, we give an immediate corollary of Theorem 4.20, using this new vocabulary.

Definition 4.21. Given $j \in \mathbb{Z}$, we write $\mathcal{D}_j = \mathcal{D}_j(\mathbb{R}^d)$ for the set of all dyadic cubes of side-length 2^j in $\mathbb{R}^d$:

$$\mathcal{D}_j = \Big\{ \prod_{l=1}^{d} \big[2^j k_l, 2^j(k_l+1)\big) \subseteq \mathbb{R}^d : k \in \mathbb{Z}^d \Big\}.$$

We also write $\mathcal{D} = \cup_j \mathcal{D}_j$. Given $Q \in \mathcal{D}$, we define f_Q by $\hat{f}_Q = \chi_Q \hat{f}$.

Corollary 4.22. *Suppose $Q, Q' \in \mathcal{D}$ with*

$$\operatorname{dist}(Q, Q') \gtrsim \operatorname{diam}(Q) = \operatorname{diam}(Q'),$$

then for some $p < 2$ (indeed, an interval of such p)

$$\big\| [e^{it\Delta} f_Q][e^{it\Delta} f_{Q'}] \big\|_{L_{t,x}^{\frac{d^2+3d+1}{d(d+1)}}} \lesssim |Q|^{1-\frac{2}{p}-\frac{1}{d^2+3d+1}} \|\hat{f}\|_{L_\xi^p(Q)} \|\hat{f}\|_{L_\xi^p(Q')}.$$

Proof. The result follows from interpolating between Theorem 4.20 and

$$\big\| [e^{it\Delta} f][e^{it\Delta} g] \big\|_{L_{t,x}^\infty} \lesssim \|\hat{f}\|_{L_\xi^1} \|\hat{g}\|_{L_\xi^1},$$

which is a consequence of the fact that the Fourier transform maps $L_\xi^1 \to L_x^\infty$. □

Our next theorem is clearly a strengthening of Theorem 4.16 (apply Hölder's inequality inside the second factor in (4.42)). The name is taken from the standard notation for the norm appearing on the right-hand side in (4.41). It was first proved in the case $d = 2$; see [**62**, Theorem 4.2]. For higher dimensions, see [**4**, Theorem 1.2] and for $d = 1$, see [**12**, Proposition 2.1].

Theorem 4.23 (X_p^q Strichartz, [**4, 12, 62**])**.** *Given $f \in \mathcal{S}$, $\frac{1}{2} < \frac{1}{p} < \frac{1}{2} + \frac{1}{(d+1)(d+2)}$, and $\frac{p}{2} < \beta < 1$,*

$$\big\| e^{it\Delta} f \big\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R}^{1+d})} \lesssim \Bigg[\sum_{Q \in \mathcal{D}} \Big(|Q|^{\frac{1}{2}-\frac{1}{p}} \big\|\hat{f}\big\|_{L_\xi^p(Q)} \Big)^{\frac{2(d+2)}{d}} \Bigg]^{\frac{d}{2(d+2)}} \tag{4.41}$$

$$\lesssim \|f\|_{L_x^2(\mathbb{R}^d)}^{\beta} \Bigg[\sup_{Q \in \mathcal{D}} |Q|^{\frac{1}{2}-\frac{1}{p}} \big\|\hat{f}\big\|_{L_\xi^p(Q)} \Bigg]^{1-\beta}. \tag{4.42}$$

Recall that this sum is over all dyadic cubes Q of all sizes.

We will not prove this result; however, the proof of Proposition 4.24 below is closely modelled on the argument given in [**4**]. This proposition is a small tweaking of (the proof of) (4.42) so as to exhibit the supremum of a spacetime norm.

Proposition 4.24. *Let $q = \frac{2(d^2+3d+1)}{d^2}$. Then*

$$\big\|e^{it\Delta} f\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R}^{1+d})} \lesssim \|f\|_{L_x^2(\mathbb{R}^d)}^{\frac{d+1}{d+2}} \bigg(\sup_{Q\in\mathcal{D}} |Q|^{\frac{d+2}{dq}-\frac12} \big\|e^{it\Delta} f_Q\big\|_{L^q_{t,x}(\mathbb{R}^{1+d})}\bigg)^{\frac{1}{d+2}}. \tag{4.43}$$

Proof. As noted above, we will be mimicking [**4**], albeit with a small twist. The first part of the argument is based on the proof of their Theorem 1.2.

Given distinct $\xi, \xi' \in \mathbb{R}^d$, there is a unique maximal pair of dyadic cubes $Q \ni \xi$ and $Q' \ni \xi'$ obeying

$$|Q| = |Q'| \quad \text{and} \quad \operatorname{dist}(Q, Q') \geq 4 \operatorname{diam}(Q). \tag{4.44}$$

Let $\mathcal{F}$ denote the family of all such pairs as $\xi \neq \xi'$ vary over $\mathbb{R}^d$. According to this definition,

$$\sum_{(Q,Q')\in\mathcal{F}} \chi_Q(\xi)\chi_{Q'}(\xi') = 1 \quad \text{for a.e. } (\xi, \xi') \in \mathbb{R}^d \times \mathbb{R}^d. \tag{4.45}$$

Note that since Q and Q' are maximal, $\operatorname{dist}(Q, Q') \leq 10 \operatorname{diam}(Q)$. In addition, this shows that given Q there are a bounded number of Q' so that $(Q, Q') \in \mathcal{F}$, that is,

$$\forall Q \in \mathcal{D}, \quad \#\big\{Q' : (Q, Q') \in \mathcal{F}\big\} \lesssim 1. \tag{4.46}$$

In view of (4.45), we can write

$$[e^{it\Delta} f]^2 = \sum_{(Q,Q')\in\mathcal{F}} [e^{it\Delta} f_Q][e^{it\Delta} f_{Q'}],$$

which clearly brings Corollary 4.22 into the game. Treating the sum via the triangle inequality is not a winning play; we need to do a bit better. The key point is to look at the spacetime Fourier supports of the products on the right-hand side. As we will see, their dilates have bounded overlap.

Given $F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ we write

$$\hat F(\omega, \xi) = (2\pi)^{-\frac{d+1}{2}} \int_{\mathbb{R}^d}\int_{\mathbb{R}} e^{i\omega t - i\xi\cdot x} F(t, x)\, dt\, dx.$$

With this convention,

$$\operatorname{supp}\big([e^{it\Delta} f_Q][e^{it\Delta} f_{Q'}]\widehat{\ }\big) \subseteq R(Q + Q') \tag{4.47}$$

where $Q+Q'$ denotes the Minkowski (or 'all pairs') sum and R denotes an associated parallelepiped that we will now define. Given a cube Q'' in $\mathbb{R}^d$ (and $Q + Q'$ is a cube), we define

$$R(Q'') = \Big\{(\omega, \eta) : \eta \in Q'' \text{ and } 2 \leq \frac{\omega - \frac12 |c(Q'')|^2 - c(Q'')\cdot[\eta - c(Q'')]}{\operatorname{diam}(Q'')^2} \leq 19\Big\}$$

where $c(Q'')$ denotes the center of the cube Q''. To verify (4.47) we merely need to note that for $\xi \in Q$ and $\xi' \in Q'$,

$$\begin{aligned} |\xi|^2 + |\xi'|^2 &= \tfrac12 |\xi + \xi'|^2 + \tfrac12 |\xi - \xi'|^2 \\ &= \tfrac12 |c(Q + Q')|^2 + c(Q + Q')\cdot[\xi + \xi' - c(Q + Q')] \end{aligned}$$

$$+ \tfrac{1}{2}|\xi + \xi' - c(Q+Q')|^2 + \tfrac{1}{2}|\xi - \xi'|^2,$$

$|\xi + \xi' - c(Q+Q')| \leq \operatorname{diam}(Q)$, and $4 \operatorname{diam}(Q) \leq |\xi - \xi'| \leq 12 \operatorname{diam}(Q)$. We also remind the reader that $\operatorname{diam}(Q + Q') = \operatorname{diam}(Q) + \operatorname{diam}(Q') = 2 \operatorname{diam}(Q)$.

Before we can turn to the analytical portion of the argument, we still need to control the overlap of the Fourier supports, or rather, of the enclosing parallelepipeds. We claim that for any $\alpha \leq 1.01$,

$$\sup_{\omega,\eta} \sum_{(Q,Q') \in \mathcal{F}} \chi_{\alpha R(Q+Q')}(\omega, \eta) \lesssim 1, \tag{4.48}$$

where αR denotes the α-dilate of R with the same center. To see this, we argue as follows: Given $(\omega, \eta) \in \alpha R(Q + Q')$, a few computations show that $\operatorname{diam}(Q)^2 \sim \omega - \frac{1}{2}|\eta|^2$, which allows us to identify the size of Q to within a bounded number of dyadic generations. This then gives an upper bound on the distance between Q and Q'. Lastly, since $\eta \in \alpha(Q + Q')$ we may deduce that both Q and Q' must lie within $O(\operatorname{diam} Q)$ of $\frac{1}{2}\eta$. To recap, each (ω, η) belongs to a bounded number of $\alpha R(Q + Q')$, which is exactly (4.48).

With the information we have gathered together, we are now ready to begin estimating the right-hand side of (4.43). For $d \geq 2$, may apply Lemma A.9, Hölder's inequality, Corollary 4.22, and (4.46) as follows:

$$\begin{aligned}
\big\| e^{it\Delta} f \big\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{2(d+2)}{d}} &= \Big\| \sum_{(Q,Q') \in \mathcal{F}} [e^{it\Delta} f_Q][e^{it\Delta} f_{Q'}] \Big\|_{L_{t,x}^{\frac{d+2}{d}}}^{\frac{d+2}{d}} \\
&\lesssim \sum_{(Q,Q') \in \mathcal{F}} \big\| [e^{it\Delta} f_Q][e^{it\Delta} f_{Q'}] \big\|_{L_{t,x}^{\frac{d+2}{d}}}^{\frac{d+2}{d}} \\
&\lesssim \sum_{(Q,Q') \in \mathcal{F}} \big\| e^{it\Delta} f_Q \big\|_{L_{t,x}^q}^{\frac{1}{d}} \big\| e^{it\Delta} f_{Q'} \big\|_{L_{t,x}^q}^{\frac{1}{d}} \big\| [e^{it\Delta} f_Q][e^{it\Delta} f_{Q'}] \big\|_{L_{t,x}^{\frac{d^2+3d+1}{d(d+1)}}}^{\frac{d+1}{d}} \\
&\lesssim \Big(\sup_{Q \in \mathcal{D}} |Q|^{\frac{d+2}{dq} - \frac{1}{2}} \big\| e^{it\Delta} f_Q \big\|_{L_{t,x}^q} \Big)^{\frac{2}{d}} \cdot \sum_{Q \in \mathcal{D}} \Big(|Q|^{-\frac{2-p}{p}} \big\| \hat{f} \big\|_{L_\xi^p(Q)}^2 \Big)^{\frac{d+1}{d}}
\end{aligned}$$

for some $p < 2$. While the final inequality obtained above holds when $d = 1$, the argument needs minor modifications (cf. the first inequality). In this case, one should use (A.2) in place of Lemma A.9; we leave the details to the reader.

In order to complete the proof of the proposition, we need to show that the sum given above can be bounded in terms of the L_ξ^2 norm of $\hat{f}$. Once again we turn to [**4**] for advice, this time, to the proof of their Theorem 1.3 (see also [**8**, p. 37] for the case $d = 2$).

The key idea is to break $\hat{f}$ into two pieces, depending on the size of Q:

$$\hat{f}(\xi) = \chi_{\{|\hat{f}| \geq 2^{-jd/2}\}}(\xi) \hat{f}(\xi) + \chi_{\{|\hat{f}| \leq 2^{-jd/2}\}}(\xi) \hat{f}(\xi) =: \hat{f}^j(\xi) + \hat{f}_j(\xi).$$

Here and below we assume (without loss of generality) that f is L_x^2-normalized; otherwise the size of f has to be incorporated into the height of this splitting, with concomitant detriment to readability.

For the first piece, we need only use the fact that $p < 2$:

$$\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j} \Big(|Q|^{-\frac{2-p}{p}} \big\| \hat{f}^j \big\|_{L_\xi^p(Q)}^2 \Big)^{\frac{d+1}{d}} \lesssim \Big(\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j} |Q|^{-\frac{2-p}{2}} \big\| \hat{f}^j \big\|_{L_\xi^p(Q)}^p \Big)^{\frac{2(d+1)}{pd}}$$

$$\lesssim \Big(\int_{\mathbb{R}^d} \sum_{j:|\hat f|\geq 2^{-jd/2}} 2^{-jd\frac{2-p}{2}} \big|\hat f(\xi)\big|^p \, d\xi \Big)^{\frac{2(d+1)}{pd}}$$
$$\lesssim \Big(\int_{\mathbb{R}^d} \big|\hat f(\xi)\big|^2 \, d\xi \Big)^{\frac{2(d+1)}{pd}} \lesssim 1.$$

For the second piece, we lead off with Hölder's inequality:

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_j} \Big(|Q|^{-\frac{2-p}{p}} \big\|\hat f_j\big\|^2_{L^p_\xi(Q)} \Big)^{\frac{d+1}{d}} &\lesssim \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_j} |Q|^{\frac1d} \big\|\hat f_j\big\|^{\frac{2(d+1)}{d}}_{L_\xi^{\frac{2(d+1)}{d}}(Q)} \\ &\lesssim \int_{\mathbb{R}^d} \sum_{j:|\hat f|\leq 2^{-jd/2}} \big(2^{-\frac{jd}{2}}\big)^{-\frac2d} \big|\hat f(\xi)\big|^{\frac{2(d+1)}{d}} \, d\xi \\ &\lesssim \int_{\mathbb{R}^d} \big|\hat f(\xi)\big|^2 \, d\xi \lesssim 1.\end{aligned}$$

This completes the proof of (4.43). □

We are now ready to state our preferred form of inverse Strichartz inequality. For other variants, see for example, [**6**, §§2–3], [**58**, Theorem 1], [**92**, Appendix A].

Proposition 4.25 (Inverse Strichartz Inequality)**.** *Fix $d \geq 1$ and $\{f_n\} \subseteq L^2_x(\mathbb{R}^d)$. Suppose that*

$$\lim_{n\to\infty} \|f_n\|_{L^2_x(\mathbb{R}^d)} = A \quad \text{and} \quad \lim_{n\to\infty} \|e^{it\Delta} f_n\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R}^{1+d})} = \varepsilon.$$

Then there exist a subsequence in n, $\phi \in L^2_x(\mathbb{R}^d)$, $\{\lambda_n\} \subseteq (0,\infty)$, $\{\xi_n\} \subseteq \mathbb{R}^d$, and $\{(t_n,x_n)\} \subseteq \mathbb{R}^{1+d}$ so that along the subsequence, we have the following:

$$\lambda_n^{\frac d2} e^{-i\xi_n\cdot(\lambda_n x + x_n)} [e^{it_n\Delta} f_n](\lambda_n x + x_n) \rightharpoonup \phi(x) \quad \text{weakly in } L^2_x(\mathbb{R}^d) \tag{4.49}$$

$$\lim_{n\to\infty} \|f_n\|^2_{L^2_x} - \|f_n - \phi_n\|^2_{L^2_x} = \|\phi\|^2_{L^2_x} \gtrsim A^2 \big(\tfrac{\varepsilon}{A}\big)^{2(d+1)(d+2)} \tag{4.50}$$

$$\limsup_{n\to\infty} \big\|e^{it\Delta}(f_n - \phi_n)\big\|^{\frac{2(d+2)}{d}}_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R}^{1+d})} \leq \varepsilon^{\frac{2(d+2)}{d}} \Big[1 - c\big(\tfrac{\varepsilon}{A}\big)^\beta\Big], \tag{4.51}$$

where c and β are (dimension-dependent) constants and

$$\phi_n(x) := e^{-it_n\Delta}[g_{0,\xi_n,x_n,\lambda_n}\phi](x) = \lambda_n^{-\frac d2} e^{-it_n\Delta}\big[e^{i\xi_n\cdot}\phi\big(\lambda_n^{-1}(\cdot - x_n)\big)\big](x). \tag{4.52}$$

Proof. By Proposition 4.24, there exists $\{Q_n\} \subseteq \mathcal{D}$ so that

$$\varepsilon^{(d+2)} A^{-(d+1)} \lesssim \liminf_{n\to\infty} |Q_n|^{\frac{d+2}{dq} - \frac12} \|e^{it\Delta}(f_n)_{Q_n}\|_{L^q_{t,x}(\mathbb{R}^{1+d})} \tag{4.53}$$

where $q = 2(d^2 + 3d + 1)/d^2$. We choose λ_n^{-1} to be the side-length of Q_n, which implies $|Q_n| = \lambda_n^{-d}$. We also set $\xi_n := c(Q_n)$, that is, the centre of this cube.

Next we determine x_n and t_n. By Hölder's inequality,

$$\begin{aligned}&\liminf_{n\to\infty} |Q_n|^{\frac{d+2}{dq} - \frac12} \|e^{it\Delta}(f_n)_{Q_n}\|_{L^q_{t,x}(\mathbb{R}^{1+d})} \\ &\quad\lesssim \liminf_{n\to\infty} |Q_n|^{\frac{d+2}{dq} - \frac12} \|e^{it\Delta}(f_n)_{Q_n}\|^{\frac{d(d+2)}{d^2+3d+1}}_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R}^{1+d})} \|e^{it\Delta}(f_n)_{Q_n}\|^{\frac{d+1}{d^2+3d+1}}_{L^\infty_{t,x}(\mathbb{R}^{1+d})} \\ &\quad\lesssim \liminf_{n\to\infty} \lambda_n^{\frac d2 - \frac{d+2}{q}} \varepsilon^{\frac{d(d+2)}{d^2+3d+1}} \|e^{it\Delta}(f_n)_{Q_n}\|^{\frac{d+1}{d^2+3d+1}}_{L^\infty_{t,x}(\mathbb{R}^{1+d})}.\end{aligned}$$

Thus by (4.53), there exists $\{(t_n, x_n)\} \subseteq \mathbb{R}^{1+d}$ so that

$$\liminf_{n\to\infty} \lambda_n^{\frac{d}{2}} \big| \big[e^{it_n\Delta}(f_n)_{Q_n}\big](x_n)\big| \gtrsim \varepsilon^{(d+1)(d+2)} A^{-(d^2+3d+1)}. \tag{4.54}$$

Having selected our symmetry parameters, weak compactness of $L_x^2(\mathbb{R}^d)$ (i.e. Alaoglu's theorem) guarantees that (4.49) holds for some $\phi \in L_x^2(\mathbb{R}^d)$ and some subsequence in n. Our next job is to show that ϕ carries non-trivial norm.

Define h so that $\hat{h}$ is the characteristic function of the cube $[-\frac{1}{2}, \frac{1}{2})^d$. From (4.54) we obtain

$$\begin{aligned} |\langle h, \phi\rangle| &= \lim_{n\to\infty} \left| \int \bar{h}(x) \lambda_n^{\frac{d}{2}} e^{-i\xi_n\cdot(\lambda_n x + x_n)} [e^{it_n\Delta} f_n](\lambda_n x + x_n)\, dx \right| \\ &= \lim_{n\to\infty} \lambda_n^{\frac{d}{2}} \big| \big[e^{it_n\Delta}(f_n)_{Q_n}\big](x_n)\big| \\ &\gtrsim \varepsilon^{(d+1)(d+2)} A^{-(d^2+3d+1)}, \end{aligned} \tag{4.55}$$

which quickly implies (4.50) as seen in the proof of Proposition 4.9. This leaves us to consider (4.51). First we claim that after passing to a subsequence,

$$e^{it\Delta}\Big[\lambda_n^{\frac{d}{2}} e^{-i\xi_n\cdot(\lambda_n x + x_n)} [e^{it_n\Delta} f_n](\lambda_n x + x_n)\Big] \to e^{it\Delta}\phi(x) \quad \text{for a.e. } (t,x) \in \mathbb{R}^{1+d}.$$

Indeed, this follows from the local smoothing estimate, Proposition 4.14, and the Rellich–Kondrashov Theorem. Thus by applying Lemma A.5 and transferring the symmetries, we obtain

$$\|e^{it\Delta} f_n\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R}^{1+d})}^{\frac{2(d+2)}{d}} - \|e^{it\Delta}(f_n - \phi_n)\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R}^{1+d})}^{\frac{2(d+2)}{d}} - \|e^{it\Delta}\phi_n\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R}^{1+d})}^{\frac{2(d+2)}{d}} \to 0.$$

The requisite lower bound on the right-hand side follows from (4.55). □

Note that one may replace (4.49) by weak convergence of the free evolutions:

Exercise. Let $\{f_n\}$ be a bounded sequence $L_x^2(\mathbb{R}^d)$. Show that $f_n \rightharpoonup f$ weakly in $L_x^2(\mathbb{R}^d)$ if and only if $e^{it\Delta} f_n \rightharpoonup e^{it\Delta} f$ weakly in $L_x^{2(d+2)/d}(\mathbb{R} \times \mathbb{R}^d)$.

The next two theorems are Strichartz analogues of the bubble decomposition discussed in the previous subsection. This kind of result was introduced by Bahouri and Gérard [**3**] in the context of the wave equation; we will follow their nomenclature and refer to it as a 'profile decomposition'. What we will present here are the mass- and energy-critical analogues of the linear profile decomposition given in that paper. Analogues of the nonlinear version appear in the proofs of Propositions 5.3 and 5.6.

The mass-critical linear profile decomposition was first proved in the case of two space dimensions. This is a result of Merle and Vega [**58**]; see also [**6**, §§2–3] for results of a very similar spirit. Carles and Keraani treated the one-dimensional case [**12**, Theorem 1.4]. The result was obtained for general dimension by Begout and Vargas [**4**]. We remind the reader that the definition of the symmetry group G associated to the mass-critical equation can be found in Subsection 2.3.

Theorem 4.26 (Mass-critical linear profile decomposition, [**4, 12, 58**])**.** *Let u_n be a bounded sequence in $L_x^2(\mathbb{R}^d)$. Then (after passing to a subsequence if necessary) there exist $J^* \in \{0, 1, \ldots\} \cup \{\infty\}$, functions $\{\phi^j\}_{j=1}^{J^*} \subseteq L_x^2(\mathbb{R}^d)$, group elements*

$\{g_n^j\}_{j=1}^{J^*} \subseteq G$, and times $\{t_n^j\}_{j=1}^{J^*} \subseteq \mathbb{R}$ so that defining w_n^J by

$$u_n = \sum_{j=1}^{J} g_n^j e^{it_n^j \Delta} \phi^j + w_n^J, \tag{4.56}$$

we have the following properties:

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|e^{it\Delta} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}} = 0 \tag{4.57}$$

$$e^{-it_n^j \Delta}\big[(g_n^j)^{-1} w_n^J\big] \rightharpoonup 0 \quad \text{weakly in } L_x^2(\mathbb{R}^d) \text{ for each } j \le J, \tag{4.58}$$

$$\sup_J \lim_{n\to\infty} \Big[\|u_n\|_{L_x^2(\mathbb{R}^d)}^2 - \sum_{j=1}^{J} \|\phi^j\|_{L_x^2(\mathbb{R}^d)}^2 - \|w_n^J\|_{L_x^2(\mathbb{R}^d)}^2\Big] = 0 \tag{4.59}$$

and lastly, for $j \neq k$ and $n \to \infty$,

$$\begin{aligned} \frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \lambda_n^j \lambda_n^k |\xi_n^j - \xi_n^k|^2 + \frac{\big|t_n^j(\lambda_n^j)^2 - t_n^k(\lambda_n^k)^2\big|}{\lambda_n^j \lambda_n^k} \\ + \frac{|x_n^j - x_n^k - 2t_n^j(\lambda_n^j)^2(\xi_n^j - \xi_n^k)|^2}{\lambda_n^j \lambda_n^k} \to \infty \end{aligned} \tag{4.60}$$

Here λ_n^j, ξ_n^j, x_n^j are the parameters associated to g_n^j (the θ parameter is zero).

Proof. Exercise: mimic the proof of Theorem 4.7 using Proposition 4.25 in place of Proposition 4.9. Note that the order of the propagator and the symmetries is changed in (4.56) relative to (4.52). As a result, the meaning of x_n^j and t_n^j has also changed relative to the parameters appearing in Proposition 4.25; indeed, the change can be deduced from

$$e^{-it_n\Delta}[g_{0,\xi_n,x_n,\lambda_n}\phi](x) = g_{t_n|\xi_n|^2,\xi_n,x_n-2t_n\xi_n,\lambda_n}\big[e^{-it_n(\lambda_n)^{-2}\Delta}\phi\big](x).$$

In addition, there is also a change in the sign of t_n^j. □

The analogue of (4.13) can be added to the conclusions of Theorem 4.26, which is to say that the profiles also decouple in the symmetric Strichartz norm; indeed, this follows *a posteriori* from (4.57) and (4.60). We will not need this fact.

The linear profile decomposition in the energy-critical case was proved by Keraani [**41**]. As in the treatment of the wave equation [**3**], the original argument used refinements of Sobolev embedding rather than of Strichartz inequality.

Theorem 4.27 (Energy-critical linear profile decomposition, [**41**])**.** *Fix $d \ge 3$ and let $\{u_n\}_{n\ge1}$ be a sequence of functions bounded in $\dot H_x^1(\mathbb{R}^d)$. Then, after passing to a subsequence if necessary, there exist $J^* \in \{0,1,\dots\} \cup \{\infty\}$, functions $\{\phi^j\}_{j=1}^{J^*} \subset \dot H_x^1(\mathbb{R}^d)$, group elements $\{g_n^j\}_{j=1}^{J^*} \subset G$, and times $\{t_n^j\}_{j=1}^{J^*} \subset \mathbb{R}$ such that for each $1 \le J \le J^*$, we have the decomposition*

$$u_n = \sum_{j=1}^{J} g_n^j e^{it_n^j \Delta} \phi^j + w_n^J \tag{4.61}$$

with the following properties:

$$\lim_{J\to J^*} \limsup_{n\to\infty} \big\|e^{it\Delta} w_n^J\big\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(\mathbb{R}\times\mathbb{R}^d)} = 0 \tag{4.62}$$

$$e^{-it_n^j \Delta}\big[(g_n^j)^{-1} w_n^J\big] \rightharpoonup 0 \quad \text{weakly in } \dot H_x^1(\mathbb{R}^d) \text{ for each } j \le J \tag{4.63}$$

$$\lim_{n\to\infty}\Big[\|\nabla u_n\|_2^2 - \sum_{j=1}^{J}\|\nabla\phi^j\|_2^2 - \|\nabla w_n^J\|_2^2\Big] = 0 \tag{4.64}$$

and for each $j \neq k$,

$$\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \frac{|x_n^j - x_n^k|^2}{\lambda_n^j\lambda_n^k} + \frac{\big|t_n^j(\lambda_n^j)^2 - t_n^k(\lambda_n^k)^2\big|}{\lambda_n^j\lambda_n^k} \to \infty \quad \text{as } n\to\infty, \tag{4.65}$$

where λ_n^j and x_n^j are the symmetry parameters associated to g_n^j by Definition 2.2; the θ parameter is identically zero.

Proof. Exercise. Deduce this result from Theorem 4.26. Note that the disappearance of the Galilei boosts can be attributed to the absence of a gradient in (4.62).

The original approach taken by Keraani involves interpolation, Theorem 4.7, and a Strichartz inequality with unequal space and time exponents. See [**41**] for more information on how this can be done. □

4.5. Radial Improvements. Most problems related to critical NLS have first been solved in the case of spherically symmetric data. This allows one to take advantage of stronger harmonic analysis tools, some of which we record below. In truth, however, the greatest advantage really appears in the nonlinear analysis.

Lemma 4.28 (Weighted Radial Strichartz, [**43**])**.** *Let $F \in L_{t,x}^{2(d+2)/(d+4)}(\mathbb{R}\times\mathbb{R}^d)$ and $u_0 \in L_x^2(\mathbb{R}^d)$ be spherically symmetric. Then,*

$$u(t) := e^{i(t-t_0)\Delta}u_0 - i\int_{t_0}^{t} e^{i(t-t')\Delta}F(t')\,dt'$$

obeys the estimate

$$\big\||x|^{\frac{2(d-1)}{q}}u\big\|_{L_t^qL_x^{\frac{2q}{q-4}}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|u_0\|_{L_x^2(\mathbb{R}^d)} + \|F\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(\mathbb{R}\times\mathbb{R}^d)}$$

for all $4 \le q \le \infty$.

Proof. For $q = \infty$, this corresponds to the trivial endpoint in the Strichartz inequality. We will only prove the result for the $q = 4$ endpoint, since the remaining cases then follow by interpolation.

As in the proof of the Strichartz inequality, the method of TT^* together with Hardy–Littlewood–Sobolev inequality reduce matters to proving that

$$\big\||x|^{\frac{d-1}{2}}e^{it\Delta}|x|^{\frac{d-1}{2}}g\big\|_{L_x^\infty(\mathbb{R}^d)} \lesssim |t|^{-\frac12}\|g\|_{L_x^1(\mathbb{R}^d)} \tag{4.66}$$

for all radial functions g.

Let P_{rad} denote the projection onto radial functions, which commutes with the free propagator. Then

$$[e^{it\Delta}P_{\text{rad}}](x,y) = (4\pi it)^{-\frac d2}e^{i\frac{|x|^2+|y|^2}{4t}}\int_{S^{d-1}} e^{-i\frac{|y|\omega\cdot x}{2t}}\,d\sigma(\omega),$$

where $d\sigma$ denotes the uniform probability measure on the unit sphere S^{d-1}. Using stationary phase (or properties of Bessel functions), one sees that

$$\big|[e^{it\Delta}P_{\text{rad}}](x,y)\big| \lesssim |t|^{-\frac d2}\big(\tfrac{|y||x|}{|t|}\big)^{-\frac{d-1}{2}} \lesssim |t|^{-\frac12}|x|^{-\frac{d-1}{2}}|y|^{-\frac{d-1}{2}}.$$

The radial dispersive estimate (4.66) now follows easily. □

The last two results are taken from the thesis work of Shuanglin Shao.

Theorem 4.29 (Shao's Strichartz Estimate, [**77**, Corollary 6.2])**.** *If $f \in L_x^2(\mathbb{R}^d)$ is spherically symmetric with $d \geq 2$, then*

$$\|P_N e^{it\Delta} f\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim_q N^{\frac{d}{2}-\frac{d+2}{q}} \|f\|_{L^2_x(\mathbb{R}^d)}, \tag{4.67}$$

provided $q > \frac{4d+2}{2d-1}$.

The new point is that q can go below $2(d+2)/d$, which is the exponent given by Theorem 4.16. The Knapp counterexample (a wave packet whose momentum is concentrated in a single direction) shows that such an improvement is not possible without the radial assumption. Spherical symmetry also allows for stronger bilinear estimates, extending both Theorem 4.18 and Theorem 4.20. We record here only a special case of [**77**, Corollary 6.5]:

Theorem 4.30 (Shao's Bilinear Estimate, [**77**, Corollary 6.5])**.** *Fix $d \geq 2$ and $f, g \in L_x^2(\mathbb{R}^d)$ spherically symmetric. Then*

$$\big\|[e^{it\Delta} f_{\leq 1}][e^{it\Delta} g_N]\big\|_{L^q_{t,x}} \lesssim N^{d-\frac{d+2}{q}} \|f\|_{L^2_x} \|g\|_{L^2_x}$$

for any $\frac{2(d+2)}{2d+1} < q \leq 2$ and $N \geq 4$.

5. Minimal blowup solutions

The purpose of this section is to prove that if the global well-posedness and scattering conjectures were to fail, then one could construct *minimal* counterexamples. These counterexamples are *minimal blowup solutions* and enjoy a wealth of properties, all of which are consequences of their minimality.

The discovery that such minimal blowup solutions would exist was made by Keraani [**42**, Theorem 1.3] in the context of the mass-critical equation. This was later adapted to the energy-critical setting by Kenig and Merle, [**38**].

We would also like to mention that earlier works on the energy-critical NLS (see [**7, 20, 75, 104**]) proposed *almost*-minimal blowup solutions as counterexamples to the global well-posedness and scattering conjecture. These solutions were shown to have space and frequency localization properties similar to (but slightly weaker than) those of the minimal blowup solutions. In fact, on a technical level, the tools involved in obtaining both types of counterexamples are closely related. However, while the earlier methods have the advantage of being quantitative, they add significantly to the complexity of the argument.

In these notes, we will only prove the existence of minimal blowup solutions for the mass- and energy-critical nonlinear Schrödinger equations. However, using the arguments presented below (especially those for the energy-critical NLS), one can construct minimal blowup solutions for the more general equation (3.5); see [**40**] for one such example.

5.1. The mass-critical NLS.

In the defocusing case, $\mu = +1$, Conjecture 1.4 says that all solutions obey spacetime bounds depending only on the mass. With this in mind, let

$$L^+(M) := \sup\{S_I(u) : \, u : I \times \mathbb{R}^d \to \mathbb{C} \text{ such that } M(u) \leq M\},$$

where the supremum is taken over all solutions $u : I \times \mathbb{R}^d \to \mathbb{C}$ to the defocusing mass-critical NLS and

$$S_I(u) := \int_I \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+2)}{d}}\, dx\, dt.$$

Note that $L^+ : [0,\infty) \to [0,\infty]$ is nondecreasing and, by Theorem 3.7, continuous. Thus, failure of Conjecture 1.4 (in the defocusing case) is equivalent to the existence of a *critical mass*, $M_c \in (0,\infty)$, so that

$$L^+(M) < \infty \quad \text{for } M < M_c \quad \text{and} \quad L^+(M) = \infty \quad \text{for } M \geq M_c.$$

Similarly, in the focusing case, $\mu = -1$, we may define $L^- : \big[0, M(Q)\big] \to [0,\infty]$ by

$$L^-(M) := \sup\{S_I(u) :\ u : I \times \mathbb{R}^d \to \mathbb{C} \text{ such that } M(u) \leq M\},$$

where the supremum is again taken over all solutions of the focusing equation. Much as before, failure of Conjecture 1.4 corresponds to the existence of a critical mass $M_c \in (0, M(Q))$, where L^- changes from being finite to infinite.

Note that the explicit solution $u(t,x) = e^{it}Q(x)$ shows that $L^-(M(Q)) = \infty$. Note also that from the local well-posedness theory (see Corollary 3.5),

$$L^+(M) + L^-(M) \lesssim M^{\frac{d+2}{d}} \quad \text{for} \quad M \leq \eta_0, \tag{5.1}$$

where $\eta_0 = \eta_0(d)$ is the threshold from the small data theory.

In order to treat the focusing and defocusing equations in as uniform a manner as possible, we adopt the following convention.

Convention. We write L for $L^\pm$ with the understanding that $L = L^+$ in the defocusing case and $L = L^-$ in the focusing case.

By the discussion above, we see that any initial data u_0 with $M(u_0) < M_c$ must give rise to a global solution, which obeys

$$S_{\mathbb{R}}(u) \leq L\big(M(u_0)\big).$$

This fact plays much the same role as the inductive hypothesis in the induction on mass/energy approach.

Our goals for this subsection are firstly, to show that if Conjecture 1.4 fails, then there exists a blowup solution u to (1.4) whose mass is exactly equal to the critical mass M_c and secondly, to derive some of its properties. In order to state the precise result, we need the following important concept:

Definition 5.1 (Almost periodicity modulo symmetries)**.** Fix μ and $d \geq 1$. A solution u to the mass-critical NLS (1.4) with lifespan I is said to be *almost periodic modulo symmetries* if there exist (possibly discontinuous) functions $N : I \to \mathbb{R}^+$, $\xi : I \to \mathbb{R}^d$, $x : I \to \mathbb{R}^d$ and a function $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\int_{|x - x(t)| \geq C(\eta)/N(t)} |u(t,x)|^2\, dx + \int_{|\xi - \xi(t)| \geq C(\eta) N(t)} |\hat{u}(t,\xi)|^2\, d\xi \leq \eta$$

for all $t \in I$ and $\eta > 0$. We refer to the function N as the *frequency scale function* for the solution u, ξ is the *frequency center function*, x is the *spatial center function*, and C is the *compactness modulus function*. Furthermore, if we can select $x(t) = \xi(t) = 0$, then we say that u is *almost periodic modulo scaling*.

Remarks. 1. The parameter $N(t)$ measures the frequency scale of the solution at time t, and $1/N(t)$ measures the spatial scale; see [**43, 96, 97**] for further discussion. Note that we have the freedom to modify $N(t)$ by any bounded function of t, provided that we also modify the compactness modulus function C accordingly. In particular, one could restrict $N(t)$ to be a power of 2 if one wished, although we will not do so here. Alternatively, the fact that the solution trajectory $t \mapsto u(t)$ is continuous in $L^2_x(\mathbb{R}^d)$ can be used to show that the functions N, ξ, x may be chosen to depend continuously on t (cf. Lemma 5.18).

2. One can view $\xi(t)$ and $x(t)$ as roughly measuring the (normalised) momentum and center-of-mass, respectively, at time t, although as u is only assumed to lie in $L^2_x(\mathbb{R}^d)$, these latter quantities are not quite rigourously defined.

3. By Proposition A.1, a family of functions is precompact in $L^2_x(\mathbb{R}^d)$ if and only if it is norm-bounded and there exists a compactness modulus function C so that

$$\int_{|x|\geq C(\eta)} |f(x)|^2 \, dx + \int_{|\xi|\geq C(\eta)} |\hat{f}(\xi)|^2 \, d\xi \leq \eta$$

for all functions f in the family. Thus, an equivalent formulation of Definition 5.1 is as follows: u is almost periodic modulo symmetries if and only if there exists a compact subset K of $L^2_x(\mathbb{R}^d)$ such that the orbit $\{u(t) : t \in I\}$ is contained inside $GK := \{gf : g \in G, f \in K\}$. This perspective also clarifies why we use the term 'almost periodic'.

We are now ready to state the main result of this subsection.

Theorem 5.2 (Reduction to almost periodic solutions, [**42, 96**])**.** *Fix μ and d and suppose that Conjecture 1.4 failed for this choice. Then there exists a maximal-lifespan solution u with mass $M(u) = M_c$, which is almost periodic modulo symmetries and which blows up both forward and backward in time.*

Remark. If we consider Conjecture 1.4 in the case of spherically symmetric data ($d \geq 2$), then the conclusion may be strengthened to almost periodicity modulo scaling, that is, $x(t) \equiv 0 \equiv \xi(t)$. This is the greatest advantage in restricting to such data.

The proof of Theorem 5.2 rests on the following key proposition, asserting a certain compactness (modulo symmetries) in sequences of solutions with mass converging to the critical mass from below.

Proposition 5.3 (Palais–Smale condition modulo symmetries, [**96**])**.** *Fix μ and d, and suppose that Conjecture 1.4 failed for this choice. Let $u_n : I_n \times \mathbb{R}^d \to \mathbb{C}$ be a sequence of solutions and $t_n \in I_n$ a sequence of times such that $M(u_n) \leq M_c$, $M(u_n) \to M_c$, and*

$$\lim_{n\to\infty} S_{\geq t_n}(u_n) = \lim_{n\to\infty} S_{\leq t_n}(u_n) = +\infty. \tag{5.2}$$

Then the sequence $Gu_n(t_n)$ has a subsequence which converges in the $G\backslash L^2_x(\mathbb{R}^d)$ topology.

Remark. The hypothesis (5.2) asserts that the sequence u_n asymptotically blows up both forward and backward in time. Both components of this hypothesis are essential, as can be seen by considering the pseudo-conformal transformation of the ground state, which only blows up in one direction (and whose orbit is non-compact in the other direction, even after quotienting out by G).

Proof. Using the time-translation symmetry of (1.4), we may take $t_n = 0$ for all n; thus, we may assume

$$\lim_{n\to\infty} S_{\geq 0}(u_n) = \lim_{n\to\infty} S_{\leq 0}(u_n) = +\infty. \tag{5.3}$$

Applying Theorem 4.26 to the bounded sequence $u_n(0)$ (passing to a subsequence if necessary), we obtain the linear profile decomposition

$$u_n(0) = \sum_{j=1}^{J} g_n^j e^{it_n^j\Delta}\phi^j + w_n^J \tag{5.4}$$

with the stated properties.

By refining the subsequence once for each j and using a standard diagonalisation argument, we may assume that for each j the sequence t_n^j, $n = 1, 2, \dots$ converges to some time $t^j \in [-\infty, +\infty]$. If $t^j \in (-\infty, +\infty)$, we may shift ϕ^j by the linear propagator $e^{it^j\Delta}$, and so assume that $t^j = 0$. Moreover, we may assume that $t_n^j \equiv 0$, since the error $e^{it_n^j\Delta}\phi^j - \phi^j$ may be absorbed into w_n^J; this will not significantly affect the scattering size of the linear evolution of w_n^J, thanks to the Strichartz inequality and the L_x^2-continuity of the free propagator. Thus, for each j either $t_n^j \equiv 0$ or $t_n^j \to \pm\infty$ as $n \to \infty$.

We now define a nonlinear profile $v^j : I^j \times \mathbb{R}^d \to \mathbb{C}$ associated to ϕ^j and depending on the limiting value of t_n^j, as follows:

- If $t_n^j \equiv 0$, we define v^j to be the maximal-lifespan solution with initial data $v^j(0) = \phi^j$.
- If $t_n^j \to \infty$, we define v^j to be the maximal-lifespan solution which scatters forward in time to $e^{it\Delta}\phi^j$.
- If $t_n^j \to -\infty$, we define v^j to be the maximal-lifespan solution which scatters backward in time to $e^{it\Delta}\phi^j$.

Finally, for each $j, n \geq 1$ we define $v_n^j : I_n^j \times \mathbb{R}^d \to \mathbb{C}$ by

$$v_n^j(t) := T_{g_n^j}\big[v^j(\cdot + t_n^j)\big](t),$$

where $I_n^j := \{t \in \mathbb{R} : (\lambda_n^j)^{-2}t + t_n^j \in I^j\}$. Each v_n^j is a solution to (1.4) with initial data $v_n^j(0) = g_n^j v^j(t_n^j)$. Note that for each J, we have

$$u_n(0) - \sum_{j=1}^{J} v_n^j(0) - w_n^J \longrightarrow 0 \quad \text{in } L_x^2 \text{ as } n \to \infty, \tag{5.5}$$

by virtue of the way v_n^j is constructed.

From Theorem 4.26 we have the mass decoupling

$$\sum_{j=1}^{J^*} M(\phi^j) \leq \limsup_{n\to\infty} M(u_n(0)) \leq M_c \tag{5.6}$$

and in particular, $\sup_j M(\phi^j) \leq M_c$.

Case I: Suppose first that

$$\sup_j M(\phi^j) \leq M_c - \varepsilon \tag{5.7}$$

for some $\varepsilon > 0$; we will eventually show that this leads to a contradiction. Indeed, by the discussion at the beginning of this subsection it follows that in this case, all

v_n^j are defined globally in time and obey the estimates

$$M(v_n^j) = M(\phi^j) \leq M_c - \varepsilon$$

and (in view of (5.1))

$$S(v_n^j) \leq L(M(\phi^j)) \lesssim M(\phi^j)^{\frac{d+2}{d}} \lesssim M(\phi^j). \tag{5.8}$$

We will eventually derive a bound on the scattering size of u_n, thus contradicting (5.3). In order to achieve this, we will use the stability result Theorem 3.7. To this end, we define an approximate solution

$$u_n^J(t) := \sum_{j=1}^{J} v_n^j(t) + e^{it\Delta} w_n^J. \tag{5.9}$$

Note that by the asymptotic orthogonality conditions in Theorem 4.26, followed by (5.8) and (5.6),

$$\begin{aligned} \lim_{J\to J^*} \limsup_{n\to\infty} S(u_n^J) &\leq \lim_{J\to J^*} \limsup_{n\to\infty} S\Bigl(\sum_{j=1}^{J} v_n^j\Bigr) \\ &= \lim_{J\to J^*} \limsup_{n\to\infty} \sum_{j=1}^{J} S(v_n^j) \lesssim \lim_{J\to J^*} \sum_{j=1}^{J} M(\phi^j) \lesssim M_c. \end{aligned} \tag{5.10}$$

We will show that u_n^J is indeed a good approximation to u_n for n, J sufficiently large.

Lemma 5.4 (Asymptotic agreement with initial data)**.** *For any $J \geq 1$ we have*

$$\lim_{n\to\infty} M\bigl(u_n^J(0) - u_n(0)\bigr) = 0.$$

PROOF. This follows from (5.5), (5.4), and (5.9). □

Lemma 5.5 (Asymptotic solution to the equation)**.** *We have*

$$\lim_{J\to J^*} \limsup_{n\to\infty} \bigl\|(i\partial_t + \Delta)u_n^J - F(u_n^J)\bigr\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(\mathbb{R}\times\mathbb{R}^d)} = 0.$$

PROOF. By the definition of u_n^J, we have

$$(i\partial_t + \Delta)u_n^J = \sum_{j=1}^{J} F(v_n^j)$$

and so, by the triangle inequality, it suffices to show that

$$\lim_{J\to J^*} \limsup_{n\to\infty} \bigl\|F(u_n^J - e^{it\Delta} w_n^J) - F(u_n^J)\bigr\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(\mathbb{R}\times\mathbb{R}^d)} = 0$$

and

$$\lim_{n\to\infty} \Bigl\| F\Bigl(\sum_{j=1}^{J} v_n^j\Bigr) - \sum_{j=1}^{J} F(v_n^j) \Bigr\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(\mathbb{R}\times\mathbb{R}^d)} = 0 \quad \text{for all } J \geq 1.$$

That the first limit is zero follows fairly quickly from the asymptotically vanishing scattering size of $e^{it\Delta} w_n^J$ together with (5.10); indeed, one need only invoke

(3.11) and Hölder's inequality. To see that the second limit is zero, we use the elementary inequality

$$\big|F\big(\sum_{j=1}^{J} z_j\big) - \sum_{j=1}^{J} F(z_j)\big| \leq C_{J,d} \sum_{j\neq j'} |z_j||z_{j'}|^{\frac{4}{d}},$$

for some $C_{J,d} < \infty$, (5.8), and the asymptotic orthogonality of the v_n^j provided by (4.60) from Theorem 4.26. □

We are now in a position to apply the stability result Theorem 3.7. Let $\delta > 0$ be a small number. Then, by the above two lemmas, we have

$$M\big(u_n^J(0) - u_n(0)\big) + \big\|(i\partial_t + \Delta)u_n^J - F(u_n^J)\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(\mathbb{R}\times\mathbb{R}^d)} \leq \delta,$$

provided J is sufficiently large (depending on δ) and n is sufficiently large (depending on J, δ). Invoking (5.10), we may apply Theorem 3.7 (for δ chosen small enough depending on M_c) to deduce that u_n exists globally and

$$S_{\mathbb{R}}(u_n) \lesssim M_c.$$

This contradicts (5.3).

Case II: The only remaining possibility is that (5.7) fails for every $\varepsilon > 0$, and thus

$$\sup_j M(\phi^j) = M_c.$$

Comparing this with (5.6), we see $J^* = 1$, that is, there is only one bubble. Consequently, the profile decomposition simplifies to

$$u_n(0) = g_n e^{it_n\Delta}\phi + w_n \tag{5.11}$$

for some sequence $t_n \in \mathbb{R}$ such that either $t_n \equiv 0$ or $t_n \to \pm\infty$, $g_n \in G$, some ϕ of mass $M(\phi) = M_c$, and some w_n with $M(w_n) \to 0$ (and hence $S(e^{it\Delta}w_n) \to 0$) as $n \to \infty$ (this is from (4.59)). By applying the symmetry operation $T_{g_n^{-1}}$ to u_n, which does not affect the hypotheses of Proposition 5.3, we may take all g_n to be the identity, and thus

$$M\big(u_n(0) - e^{it_n\Delta}\phi\big) \to 0 \text{ as } n \to \infty.$$

If $t_n \equiv 0$, then $u_n(0)$ converge in $L_x^2(\mathbb{R}^d)$ to ϕ, and thus $Gu_n(0)$ converge in $G\backslash L_x^2(\mathbb{R}^d)$, as desired. So the only remaining case is when $t_n \to \pm\infty$; we shall assume that $t_n \to \infty$, as the other case is similar. By the Strichartz inequality we have

$$S_{\mathbb{R}}(e^{it\Delta}\phi) < \infty$$

and hence, by time-translation invariance and monotone convergence,

$$\lim_{n\to\infty} S_{\geq 0}(e^{it\Delta}e^{it_n\Delta}\phi) = 0.$$

As the action of G preserves linear solutions of the Schrödinger equation, we have $e^{it\Delta}g_n = T_{g_n}e^{it\Delta}$; as T_{g_n} preserves the scattering norm S (as well as $S_{\geq 0}$ and $S_{\leq 0}$), we deduce

$$\lim_{n\to\infty} S_{\geq 0}(e^{it\Delta}g_n e^{it_n\Delta}\phi) = 0.$$

Since $S(e^{it\Delta}w_n) \to 0$ as $n \to \infty$, we see from (5.11) that

$$\lim_{n\to\infty} S_{\geq 0}(e^{it\Delta}u_n(0)) = 0.$$

Applying Theorem 3.7 (using 0 as the approximate solution and $u_n(0)$ as the initial data), we conclude that

$$\lim_{n\to\infty} S_{\geq 0}(u_n) = 0.$$

But this contradicts one of the estimates in (5.3). A similar argument, using the other half of (5.3), allows us to exclude the possibility that $t_n \to -\infty$. This concludes the proof of Proposition 5.3. □

We are finally ready to extract the minimal-mass blowup solution to (1.4).

PROOF OF THEOREM 5.2. By the definition of the critical mass M_c (and the continuity of L), we can find a sequence $u_n : I_n \times \mathbb{R}^d \to \mathbb{C}$ of maximal-lifespan solutions with $M(u_n) \leq M_c$ and $\lim_{n\to\infty} S(u_n) = +\infty$. By choosing $t_n \in I_n$ to be the median time of the $L_{t,x}^{2(d+2)/d}$ norm of u_n (cf. the "middle third" trick in [**7**]), we can thus arrange that (5.2) holds. By time-translation invariance we may take $t_n = 0$.

Invoking Proposition 5.3 and passing to a subsequence if necessary, we find group elements $g_n \in G$ such that $g_n u_n(0)$ converges strongly in $L_x^2(\mathbb{R}^d)$ to some function $u_0 \in L_x^2(\mathbb{R}^d)$. By applying the group action T_{g_n} to the solutions u_n we may take g_n to all be the identity; thus, $u_n(0)$ converge strongly in $L_x^2(\mathbb{R}^d)$ to u_0. In particular this implies $M(u_0) \leq M_c$.

Let $u : I \times \mathbb{R}^n \to \mathbb{C}$ be the maximal-lifespan solution to (1.4) with initial data $u(0) = u_0$ as given by Corollary 3.5. We claim that u blows up both forward and backward in time. Indeed, if u does not blow up forward in time (say), then $[0, +\infty) \subseteq I$ and $S_{\geq 0}(u) < \infty$. By Theorem 3.7, this implies that for sufficiently large n, we have $[0, +\infty) \subseteq I_n$ and

$$\limsup_{n\to\infty} S_{\geq 0}(u_n) < \infty,$$

contradicting (5.2). By the definition of M_c, this forces $M(u_0) \geq M_c$ and hence $M(u_0)$ must be exactly M_c.

It remains to show that the solution u is almost periodic modulo G. Consider an arbitrary sequence $u(t'_n)$ in the orbit $\{u(t) : t \in I\}$. Now, since u blows up both forward and backward in time, but is locally in $L_{t,x}^{2(d+2)/d}$, we have

$$S_{\geq t'_n}(u) = S_{\leq t'_n}(u) = \infty.$$

Applying Proposition 5.3 once again, we see that $Gu(t'_n)$ has a convergent subsequence in $G\backslash L_x^2(\mathbb{R}^d)$. Thus, the orbit $\{Gu(t) : t \in I\}$ is precompact in $G\backslash L_x^2(\mathbb{R}^d)$, as desired. □

5.2. The energy-critical NLS. In this subsection, we outline the proof of the existence of a minimal kinetic energy blowup solution to the energy-critical NLS (1.6). The argument we present is from [**44**], which builds upon earlier work by Kenig and Merle [**38**]. The fact that the kinetic energy is not a conserved quantity for (1.6) introduces several difficulties over the material presented in the previous subsection. We will elaborate upon them at the appropriate time.

Let us start by investigating what the failure of Conjecture 1.5 would imply.

If $\mu = +1$, for any $0 \leq E_0 < \infty$, we define

$$L^+(E_0) := \sup\{S_I(u) : \ u : I \times \mathbb{R}^d \to \mathbb{C} \text{ such that } \sup_{t\in I} \|\nabla u(t)\|_2^2 \leq E_0\},$$

where the supremum is taken over all solutions $u : I\times\mathbb{R}^d \to \mathbb{C}$ to (1.6). Throughout this subsection we will use the notation

$$S_I(u) := \int_I \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+2)}{d-2}}\,dx\,dt$$

for the scattering size of u on an interval I. Note that this is an energy-critical Strichartz norm.

Similarly, if $\mu = -1$, for any $0 \le E_0 \le \|\nabla W\|_2^2$, we define

$$L^-(E_0) := \sup\{S_I(u) : u : I \times \mathbb{R}^d \to \mathbb{C} \text{ such that } \sup_{t\in I} \|\nabla u(t)\|_2^2 \le E_0\},$$

where the supremum is again taken over all solutions $u : I \times \mathbb{R}^d \to \mathbb{C}$ to (1.6).

Thus, $L^+ : [0,\infty) \to [0,\infty]$ and $L^- : \big[0, \|\nabla W\|_2^2\big] \to [0,\infty]$ are non-decreasing functions with $L^-\big(\|\nabla W\|_2^2\big) = \infty$. Moreover, from the local well-posedness theory (see Corollary 3.5),

$$L^+(E_0) + L^-(E_0) \lesssim E_0^{\frac{d+2}{d-2}} \quad \text{for} \quad E_0 \le \eta_0,$$

where $\eta_0 = \eta_0(d)$ is the threshold from the small data theory.

From the stability result Theorem 3.8, we see that L^+ and L^- are continuous. Therefore, there must exist a unique *critical kinetic energy* E_c such that $0 < E_c \le \infty$ if $\mu > 0$ and $0 < E_c \le \|\nabla W\|_2^2$ if $\mu < 0$ and such that $L^\pm(E_0) < \infty$ for $E_0 < E_c$ and $L^\pm(E_0) = \infty$ for $E_0 \ge E_c$. To ease notation, we adopt the same convention as in the mass-critical case:

Convention. We write L for $L^\pm$ with the understanding that $L = L^+$ in the defocusing case and $L = L^-$ in the focusing case.

By the discussion above, we see that if $u : I \times \mathbb{R}^d \to \mathbb{C}$ is a maximal-lifespan solution to (1.6) such that $\sup_{t\in I} \|\nabla u(t)\|_2^2 < E_c$, then u is global and

$$S_{\mathbb{R}}(u) \le L\big(\sup_{t\in I} \|\nabla u(t)\|_2^2\big).$$

Failure of Conjecture 1.5 is equivalent to $0 < E_c < \infty$ in the defocusing case and $0 < E_c < \|\nabla W\|_2^2$ in the focusing case.

Just as in the mass-critical case, the extraction of a minimal blowup solution will be a consequence of the following key compactness result.

Proposition 5.6 (Palais–Smale condition modulo symmetries, [**44**]). *Fix μ and $d \ge 3$. Let $u_n : I_n \times \mathbb{R}^d \mapsto \mathbb{C}$ be a sequence of solutions to* (1.6) *such that*

$$\limsup_{n\to\infty} \sup_{t\in I_n} \|\nabla u_n(t)\|_2^2 = E_c \tag{5.12}$$

and

$$\lim_{n\to\infty} S_{\ge t_n}(u_n) = \lim_{n\to\infty} S_{\le t_n}(u_n) = \infty.$$

for some sequence of times $t_n \in I_n$. Then the sequence $u_n(t_n)$ has a subsequence which converges in $\dot H^1_x(\mathbb{R}^d)$ modulo symmetries.

Proof. Using the time-translation symmetry of the equation (1.6), we may set $t_n = 0$ for all $n \ge 1$. Thus,

$$\lim_{n\to\infty} S_{\ge 0}(u_n) = \lim_{n\to\infty} S_{\le 0}(u_n) = \infty. \tag{5.13}$$

Applying the linear profile decomposition Theorem 4.27 to the sequence $u_n(0)$ (which is bounded in $\dot H^1_x(\mathbb{R}^d)$ by (5.12)) and passing to a subsequence if necessary, we obtain the decomposition

$$u_n(0) = \sum_{j=1}^{J} g_n^j e^{it_n^j\Delta}\phi^j + w_n^J.$$

Arguing as in the proof of Proposition 5.3, we may assume that for each $j \geq 1$ either $t_n^j \equiv 0$ or $t_n^j \to \pm\infty$ as $n \to \infty$. Continuing as there, we define the nonlinear profiles $v^j : I^j \times \mathbb{R}^d \to \mathbb{C}$ and $v_n^j : I_n^j \times \mathbb{R}^d \to \mathbb{C}$.

By the asymptotic decoupling of the kinetic energy, there exists $J_0 \geq 1$ such that

$$\|\nabla\phi^j\|_2^2 \leq \eta_0 \quad \text{for all} \quad j \geq J_0,$$

where $\eta_0 = \eta_0(d)$ is the threshold for the small data theory. Hence, by Corollary 3.9, for all $n \geq 1$ and all $j \geq J_0$ the solutions v_n^j are global and moreover,

$$\sup_{t\in\mathbb{R}} \|\nabla v_n^j(t)\|_2^2 + S_{\mathbb{R}}(v_n^j) \lesssim \|\nabla\phi^j\|_2^2. \tag{5.14}$$

At this point the proof of the Palais–Smale condition for the energy-critical NLS starts to diverge from that in the mass-critical case. Indeed, as the kinetic energy is not a conserved quantity, even if $v_n^j(0) = g_n^j v^j(t_n^j)$ has kinetic energy less than the critical value E_c, this does not guarantee the same will hold throughout the lifespan of v_n^j and in particular, it does not guarantee global existence nor global spacetime bounds. As a consequence, we must actively search for a profile responsible for the asymptotic blowup (5.13). As we will see shortly, the existence of at least one such profile is a consequence of the stability result Theorem 3.8 and the asymptotic orthogonality of the profiles given by Theorem 4.27.

Lemma 5.7 (At least one bad profile)**.** *There exists $1 \leq j_0 < J_0$ such that*

$$\limsup_{n\to\infty} S_{[0,\,\sup I_n^{j_0})}(v_n^{j_0}) = \infty.$$

Proof. We argue by contradiction. Assume that for all $1 \leq j < J_0$,

$$\limsup_{n\to\infty} S_{[0,\,\sup I_n^{j})}(v_n^{j}) < \infty. \tag{5.15}$$

In particular, this implies $\sup I_n^j = \infty$ for all $1 \leq j < J_0$ and all sufficiently large n. Combining (5.15) with (5.14), and then using (5.12),

$$\sum_{j\geq 1} S_{[0,\infty)}(v_n^j) \lesssim 1 + \sum_{j\geq J_0} \|\nabla\phi^j\|_2^2 \lesssim 1 + E_c \tag{5.16}$$

for all n sufficiently large.

Using the estimates above and the stability result Theorem 3.8, we will derive a bound on the scattering size of u_n (for n sufficiently large), thus contradicting (5.13). To this end, we define the approximate solution

$$u_n^J(t) := \sum_{j=1}^{J} v_n^j(t) + e^{it\Delta} w_n^J.$$

Note that by (5.16) and the asymptotic vanishing of the scattering size of $e^{it\Delta}w_n^J$,

$$\begin{aligned}\lim_{J\to J^*}\limsup_{n\to\infty} S_{[0,\infty)}(u_n^J) &\lesssim \lim_{J\to J^*}\limsup_{n\to\infty}\Big(S_{[0,\infty)}\big(\sum_{j=1}^J v_n^j\big) + S_{[0,\infty)}\big(e^{it\Delta}w_n^J\big)\Big)\\ &\lesssim \lim_{J\to J^*}\limsup_{n\to\infty}\sum_{j=1}^J S_{[0,\infty)}(v_n^j) \lesssim 1+E_c.\end{aligned} \tag{5.17}$$

The next two lemmas show that u_n^J is indeed a good approximation to u_n for n and J sufficiently large.

Lemma 5.8 (Asymptotic agreement with initial data)**.** *For any $J\geq 1$ we have*

$$\lim_{n\to\infty}\big\|u_n^J(0)-u_n(0)\big\|_{\dot H_x^1(\mathbb{R}^d)}=0.$$

Proof. Exercise: mimic the proof of Lemma 5.4. ☐

Lemma 5.9 (Asymptotic solution to the equation)**.** *We have*

$$\lim_{J\to J^*}\limsup_{n\to\infty}\big\|\nabla\big[(i\partial_t+\Delta)u_n^J - F(u_n^J)\big]\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([0,\infty)\times\mathbb{R}^d)}=0.$$

Proof. Exercise: mimic the proof of Lemma 5.5. There is one new difficulty, namely, one needs to show that

$$\lim_{J\to J^*}\limsup_{n\to\infty}\|v_n^j\nabla e^{it\Delta}w_n^J\|_{L_{t,x}^{\frac{d+2}{d-1}}([0,\infty)\times\mathbb{R}^d)}=0$$

for each $j\leq J$. After transferring symmetries to w_n^J, this follows from Corollary 4.15. ☐

We are now in a position to apply the stability result Theorem 3.8. Indeed, invoking the two lemmas above and (5.17), we conclude that for n sufficiently large,

$$S_{[0,\infty)}(u_n)\lesssim 1+E_c,$$

thus contradicting (5.13). This finishes the proof of Lemma 5.7. ☐

Returning to the proof of Proposition 5.6 and rearranging the indices, we may assume that there exists $1\leq J_1<J_0$ such that

$$\limsup_{n\to\infty} S_{[0,\sup I_n^j)}(v_n^j)=\infty \text{ for } 1\leq j\leq J_1 \quad\text{and}\quad \limsup_{n\to\infty} S_{[0,\infty)}(v_n^j)<\infty \text{ for } j>J_1.$$

Passing to a subsequence in n, we can guarantee that $S_{[0,\sup I_n^1)}(v_n^1)\to\infty$.

At this point our enemy scenario is that consisting of two or more profiles that take turns at driving the scattering norm of u_n to infinity. In order to finish the proof of the Palais–Smale condition, we have to prove that only one profile is responsible for the asymptotic blowup (5.13). In order to achieve this, we have to prove kinetic energy decoupling for the nonlinear profiles for large periods of time, large enough that the kinetic energy of v_n^1 has achieved the critical kinetic energy.

For each $m,n\geq 1$ let us define an integer $j(m,n)\in\{1,\dots,J_1\}$ and an interval K_n^m of the form $[0,\tau]$ by

$$\sup_{1\leq j\leq J_1} S_{K_n^m}(v_n^j)=S_{K_n^m}(v_n^{j(m,n)})=m. \tag{5.18}$$

By the pigeonhole principle, there is a $1\leq j_1\leq J_1$ so that for infinitely many m one has $j(m,n)=j_1$ for infinitely many n. Note that the infinite set of n for which this holds may be m-dependent. By reordering the indices, we may assume that $j_1 = 1$. Then, by the definition of the critical kinetic energy, we obtain

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \sup_{t\in K_n^m} \|\nabla v_n^1(t)\|_2^2 \geq E_c. \tag{5.19}$$

On the other hand, by virtue of (5.18), all v_n^j have finite scattering size on K_n^m for each $m \geq 1$. Thus, by the same argument used in Lemma 5.7, we see that for n and J sufficiently large, u_n^J is a good approximation to u_n on each K_n^m. More precisely,

$$\lim_{J\to J^*} \limsup_{n\to\infty} \|u_n^J - u_n\|_{L_t^\infty \dot H_x^1(K_n^m\times\mathbb{R}^d)} = 0 \tag{5.20}$$

for each $m \geq 1$.

Our next result proves asymptotic kinetic energy decoupling for u_n^J.

Lemma 5.10 (Kinetic energy decoupling for u_n^J)**.** *For all $J \geq 1$ and $m \geq 1$,*

$$\limsup_{n\to\infty} \sup_{t\in K_n^m} \Big| \|\nabla u_n^J(t)\|_2^2 - \sum_{j=1}^J \|\nabla v_n^j(t)\|_2^2 - \|\nabla w_n^J\|_2^2 \Big| = 0.$$

Proof. Fix $J \geq 1$ and $m \geq 1$. Then, for all $t \in K_n^m$,

$$\begin{aligned} \|\nabla u_n^J(t)\|_2^2 &= \langle \nabla u_n^J(t), \nabla u_n^J(t)\rangle \\ &= \sum_{j=1}^J \|\nabla v_n^j(t)\|_2^2 + \|\nabla w_n^J\|_2^2 + \sum_{j\neq j'} \langle \nabla v_n^j(t), \nabla v_n^{j'}(t)\rangle \\ &\quad + \sum_{j=1}^J \big(\big\langle \nabla e^{it\Delta} w_n^J, \nabla v_n^j(t)\big\rangle + \big\langle \nabla v_n^j(t), \nabla e^{it\Delta} w_n^J\big\rangle\big). \end{aligned}$$

To prove Lemma 5.10, it thus suffices to show that for all sequences $t_n \in K_n^m$,

$$\langle \nabla v_n^j(t_n), \nabla v_n^{j'}(t_n)\rangle \to 0 \quad \text{as } n\to\infty \tag{5.21}$$

and

$$\big\langle \nabla e^{it_n\Delta} w_n^J, \nabla v_n^j(t_n)\big\rangle \to 0 \quad \text{as } n\to\infty \tag{5.22}$$

for all $1 \leq j, j' \leq J$ with $j \neq j'$. We will only demonstrate the latter, which requires (4.63); the former can be deduced in much the same manner using the asymptotic orthogonality of the nonlinear profiles.

By a change of variables,

$$\big\langle \nabla e^{it_n\Delta} w_n^J, \nabla v_n^j(t_n)\big\rangle = \big\langle \nabla e^{it_n(\lambda_n^j)^{-2}\Delta}[(g_n^j)^{-1} w_n^J], \nabla v^j\big(\tfrac{t_n}{(\lambda_n^j)^2} + t_n^j\big)\big\rangle. \tag{5.23}$$

As $t_n \in K_n^m \subseteq [0, \sup I_n^j)$ for all $1 \leq j \leq J_1$, we have $t_n(\lambda_n^j)^{-2} + t_n^j \in I^j$ for all $j \geq 1$. Recall that I^j is the maximal lifespan of v^j; for $j > J_1$ this is $\mathbb{R}$. By refining the sequence once for every j and using the standard diagonalisation argument, we may assume $t_n(\lambda_n^j)^{-2} + t_n^j$ converges for every j.

Fix $1 \leq j \leq J$. If $t_n(\lambda_n^j)^{-2} + t_n^j$ converges to some point τ^j in the interior of I^j, then by the continuity of the flow, $v^j\big(t_n(\lambda_n^j)^{-2} + t_n^j\big)$ converges to $v^j(\tau^j)$ in $\dot H_x^1(\mathbb{R}^d)$. On the other hand,

$$\limsup_{n\to\infty} \big\| e^{it_n(\lambda_n^j)^{-2}\Delta}[(g_n^j)^{-1} w_n^J]\big\|_{\dot H_x^1(\mathbb{R}^d)} = \limsup_{n\to\infty} \|w_n^J\|_{\dot H_x^1(\mathbb{R}^d)} \lesssim E_c. \tag{5.24}$$

Combining this with (5.23), we obtain

$$\begin{aligned}\lim_{n\to\infty}\big\langle \nabla e^{it_n\Delta}w_n^J, \nabla v_n^j(t_n)\big\rangle &= \lim_{n\to\infty}\big\langle \nabla e^{it_n(\lambda_n^j)^{-2}\Delta}[(g_n^j)^{-1}w_n^J], \nabla v^j(\tau^j)\big\rangle\\ &= \lim_{n\to\infty}\big\langle \nabla e^{-it_n^j\Delta}[(g_n^j)^{-1}w_n^J], \nabla e^{-i\tau^j\Delta}v^j(\tau^j)\big\rangle.\end{aligned}$$

Invoking (4.63), we deduce (5.22).

Consider now the case when $t_n(\lambda_n^j)^{-2}+t_n^j$ converges to $\sup I^j$. Then we must have $\sup I^j=\infty$ and v^j scatters forward in time. This is clearly true if $t_n^j\to\infty$ as $n\to\infty$; in the other cases, failure would imply

$$\limsup_{n\to\infty} S_{[0,t_n]}(v_n^j) = \limsup_{n\to\infty} S_{\left[t_n^j, t_n(\lambda_n^j)^{-2}+t_n^j\right]}(v^j)=\infty,$$

which contradicts $t_n\in K_n^m$. Therefore, there exists $\phi^j\in \dot H_x^1(\mathbb{R}^d)$ such that

$$\lim_{n\to\infty}\Big\| v^j\big(t_n(\lambda_n^j)^{-2}+t_n^j\big) - e^{i\left(t_n(\lambda_n^j)^{-2}+t_n^j\right)\Delta}\phi^j\Big\|_{\dot H_x^1(\mathbb{R}^d)}=0.$$

Together with (5.23), this yields

$$\lim_{n\to\infty}\big\langle \nabla e^{it_n\Delta}w_n^J, \nabla v_n^j(t_n)\big\rangle = \lim_{n\to\infty}\big\langle \nabla e^{-it_n^j\Delta}[(g_n^j)^{-1}w_n^J], \nabla\phi^j\big\rangle,$$

which by (4.63) implies (5.22).

Finally, we consider the case when $t_n(\lambda_n^j)^{-2}+t_n^j$ converges to $\inf I^j$. Since $t_n(\lambda_n^j)^{-2}\geq 0$ and $\inf I^j<\infty$ for all $j\geq 1$ we see that t_n^j does not converge to $+\infty$. Moreover, if $t_n^j\equiv 0$, then $\inf I^j<0$; as $t_n(\lambda_n^j)^{-2}\geq 0$, we see that t_n^j cannot be identically zero. This leaves $t_n^j\to-\infty$ as $n\to\infty$. Thus $\inf I^j=-\infty$ and v^j scatters backward in time to $e^{it\Delta}\phi^j$. We obtain

$$\lim_{n\to\infty}\Big\| v^j\big(t_n(\lambda_n^j)^{-2}+t_n^j\big) - e^{i\left(t_n(\lambda_n^j)^{-2}+t_n^j\right)\Delta}\phi^j\Big\|_{\dot H_x^1(\mathbb{R}^d)}=0,$$

which by (5.23) implies

$$\lim_{n\to\infty}\big\langle \nabla e^{it_n\Delta}w_n^J, \nabla v_n^j(t_n)\big\rangle = \lim_{n\to\infty}\big\langle \nabla e^{-it_n^j\Delta}[(g_n^j)^{-1}w_n^J], \nabla\phi^j\big\rangle.$$

Invoking (4.63) once again, we derive (5.22).

This finishes the proof of Lemma 5.10. □

Returning to the proof of Proposition 5.6 and using (5.12) and (5.20) together with Lemma 5.10, we find

$$E_c \geq \limsup_{n\to\infty}\sup_{t\in K_n^m}\|\nabla u_n(t)\|_2^2 = \lim_{J\to\infty}\limsup_{n\to\infty}\Big\{\|\nabla w_n^J\|_2^2 + \sup_{t\in K_n^m}\sum_{j=1}^J\|\nabla v_n^j(t)\|_2^2\Big\}$$

for each $m\geq 1$. Invoking (5.19), we thus obtain the simplified decomposition

$$u_n(0)=g_n e^{i\tau_n\Delta}\phi + w_n \tag{5.25}$$

for some $g_n\in G$, $\tau_n\in\mathbb{R}$, and some functions $\phi, w_n\in\dot H_x^1(\mathbb{R}^d)$ with $w_n\to 0$ strongly in $\dot H_x^1(\mathbb{R}^d)$. Moreover, the sequence τ_n obeys $\tau_n\equiv 0$ or $\tau_n\to\pm\infty$.

If $\tau_n\equiv 0$, (5.25) immediately implies that $u_n(0)$ converge modulo symmetries to ϕ, which proves Proposition 5.6 in this case. Finally, arguing as in the proof of the Palais–Smale condition in the mass-critical case, one shows that this is the only possible case, that is, τ_n cannot converge to either ∞ or $-\infty$.

This completes the proof of Proposition 5.6. □

With the Palais–Smale condition in place, we can now extract a minimal blowup solution, very much as we did in the previous subsection. Let us first revisit the definition of almost periodicity in the energy-critical context.

Definition 5.11 (Almost periodicity modulo symmetries)**.** Fix μ and $d \geq 3$. A solution u to the energy-critical NLS (1.6) with lifespan I and uniformly bounded kinetic energy is said to be *almost periodic modulo symmetries* if there exist (possibly discontinuous) functions $N : I \to \mathbb{R}^+$, $x : I \to \mathbb{R}^d$, and a function $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\int_{|x-x(t)|\geq C(\eta)/N(t)} |\nabla u(t,x)|^2 \, dx + \int_{|\xi|\geq C(\eta)N(t)} |\xi \hat{u}(t,\xi)|^2 \, d\xi \leq \eta$$

for all $t \in I$ and $\eta > 0$. We refer to the function N as the *frequency scale function* for the solution u, x is the *spatial center function*, and C is the *compactness modulus function*.

Remark. Comparing Definitions 5.1 and 5.11, we see that there are two differences. The first is that in the energy-critical case, compactness is in $\dot{H}^1_x$ rather than in L^2_x. A deeper difference is the absence of Galilei boosts among the symmetry parameters in the energy-critical case. While Galilei boosts leave the mass and the equation invariant, they modify the energy (cf. Proposition 2.3); boundedness of the kinetic energy implies $|\xi(t)|/N(t) = O(1)$, which allows us to take $\xi(t) \equiv 0$ in the definition above, modifying the compactness modulus function if necessary.

We are now ready to introduce the central result of this subsection.

Theorem 5.12 (Reduction to almost periodic solutions, [**44**])**.** *Fix μ and $d \geq 3$ and suppose that Conjecture 1.5 failed for this choice of μ and d. Then there exists a maximal-lifespan solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ to* (1.6) *such that* $\sup_{t\in I} \|\nabla u(t)\|_2^2 = E_c$, *$u$ is almost periodic modulo symmetries and blows up both forward and backward in time.*

Proof. Exercise. □

5.3. Almost periodic solutions. In this subsection, we continue our study of solutions to (1.4) and (1.6) that are almost periodic modulo symmetries. We record basic properties of the frequency scale function $N(t)$, spatial center function $x(t)$, and frequency center function $\xi(t)$. Most of the material we present is taken from [**43**].

Lemma 5.13 (Quasi-uniqueness of $N(t), x(t), \xi(t)$)**.** *Let u be a non-zero solution to* (1.4) *with lifespan I, which is almost periodic modulo symmetries with parameters $N(t), x(t), \xi(t)$ and compactness modulus function C, and also almost periodic modulo symmetries with parameters $N'(t), x'(t), \xi'(t)$ and compactness modulus function C'. Then we have*

$$N(t) \sim_{u,C,C'} N'(t), \quad |x(t) - x'(t)| \lesssim_{u,C,C'} \frac{1}{N(t)}, \quad |\xi(t) - \xi'(t)| \lesssim_{u,C,C'} N(t)$$

for all $t \in I$. A similar result holds for almost periodic solutions to (1.6).

Proof. Let u be a solution to (1.4). We turn to the first claim and notice that by symmetry, it suffices to establish the bound $N'(t) \lesssim_{u,C,C'} N(t)$.

Fix t and let $\eta > 0$ to be chosen later. By Definition 5.1 we have

$$\int_{|x-x'(t)|\geq C'(\eta)/N'(t)} |u(t,x)|^2\,dx \leq \eta$$

and

$$\int_{|\xi-\xi(t)|\geq C(\eta)N(t)} |\hat{u}(t,\xi)|^2\,d\xi \leq \eta.$$

We split $u := u_1 + u_2$, where $u_1(t,x) := u(t,x)\chi_{|x-x'(t)|\geq C'(\eta)/N'(t)}$ and $u_2(t,x) := u(t,x)\chi_{|x-x'(t)|<C'(\eta)/N'(t)}$. Then, by Plancherel's theorem we have

$$\int_{\mathbb{R}^d} |\hat{u}_1(t,\xi)|^2\,d\xi \lesssim \eta, \tag{5.26}$$

while from Cauchy-Schwarz we have

$$\sup_{\xi\in\mathbb{R}^d} |\hat{u}_2(t,\xi)|^2 \lesssim_{\eta,C'} M(u)N'(t)^{-d}.$$

Integrating the last inequality over the ball $|\xi - \xi(t)| \leq C(\eta)N(t)$ and invoking (5.26), we conclude that

$$\int_{\mathbb{R}^d} |\hat{u}(t,\xi)|^2\,d\xi \lesssim \eta + O_{\eta,C,C'}(M(u)N(t)^d N'(t)^{-d}).$$

Thus, by Plancherel and mass conservation,

$$M(u) \lesssim \eta + O_{\eta,C,C'}(M(u)N(t)^d N'(t)^{-d}).$$

Choosing η to be a small multiple of $M(u)$ (which is non-zero by hypothesis), we obtain the first claim.

The last two claims now follow from a quick inspection of Definition 5.1. □

To describe how the symmetry parameters depend on u, we use the natural notion of convergence for solutions:

Definition 5.14 (Convergence of solutions)**.** Let $u_n : I_n \times \mathbb{R}^d \to \mathbb{C}$ be a sequence of solutions to the mass-critical NLS, let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be another solution, and let K be a compact time interval. We say that u_n *converge uniformly to* u on K if $K \subset I$, $K \subset I_n$ for all sufficiently large n, and u_n converges strongly to u in $C_t^0 L_x^2(K \times \mathbb{R}^d) \cap L_{t,x}^{2(d+2)/d}(K \times \mathbb{R}^d)$ as $n \to \infty$. We say that u_n *converge locally uniformly to* u if u_n converges uniformly to u on every compact interval $K \subset I$.

In the energy-critical case, we ask that $u_n \to u$ on $K \times \mathbb{R}^d$ in the $C_t^0 \dot{H}_x^1 \cap L_{t,x}^{2(d+2)/(d-2)}$ topology.

Lemma 5.15 (Quasi-continuous dependence of $N(t), x(t), \xi(t)$ on u)**.** *Let u_n be a sequence of solutions to* (1.4) *with lifespans I_n, which are almost periodic modulo symmetries with parameters $N_n(t), x_n(t), \xi_n(t)$ and compactness modulus function $C : \mathbb{R}^+ \to \mathbb{R}^+$, independent of n. Suppose that u_n converge locally uniformly to a non-zero solution u to* (1.4) *with lifespan I. Then u is almost periodic modulo symmetries with some parameters $N(t), x(t), \xi(t)$ and the same compactness modulus function C. Furthermore, we have*

$$\liminf_{n\to\infty} N_n(t) \lesssim_{u,C} N(t) \lesssim_{u,C} \limsup_{n\to\infty} N_n(t) \tag{5.27}$$

$$\limsup_{n\to\infty} |x_n(t) - x(t)| \lesssim_{u,C} \frac{1}{N(t)} \tag{5.28}$$

$$\limsup_{n\to\infty} |\xi_n(t) - \xi(t)| \lesssim_{u,C} N(t) \tag{5.29}$$

for all $t \in I$. A similar result holds for the energy-critical NLS.

Proof. We first show that

$$0 < \liminf_{n\to\infty} N_n(t) \le \limsup_{n\to\infty} N_n(t) < \infty \tag{5.30}$$

$$\limsup_{n\to\infty} |x_n(t)| N_n(t) + \limsup_{n\to\infty} \frac{|\xi_n(t)|}{N_n(t)} < \infty \tag{5.31}$$

for all $t \in I$. Indeed, if one of the inequalities in (5.30) failed for some t, then (by passing to a subsequence if necessary) $N_n(t)$ would converge to zero or to infinity as $n \to \infty$. Thus, by Definition 5.1, $u_n(t)$ would converge weakly to zero, and hence, by the local uniform convergence, would converge strongly to zero. But this contradicts the hypothesis that u is not identically zero. This establishes (5.30). A similar argument settles (5.31).

From (5.30) and (5.31), we see that for each $t \in I$ the sequences $N_n(t)$, $x_n(t)$, and $\xi_n(t)$ each have at least one limit point, which we denote $N(t)$, $x(t)$, and $\xi(t)$, respectively. Using the local uniform convergence, we easily verify that u is almost periodic modulo symmetries with parameters $N(t), x(t), \xi(t)$ and compactness modulus function C.

It remains to establish (5.27) through (5.29), which we prove by contradiction. Suppose for example that (5.27) failed. Then given any A, there exists a $t \in I$ for which $N_n(t)$ has at least two limit points which are separated by a ratio of at least A, and so u has two frequency scale functions with compactness modulus function C, which are separated by this ratio. This contradicts Lemma 5.13 for A large enough depending on u. Hence (5.27) holds. A similar argument establishes (5.28) and (5.29). □

Definition 5.16 (Normalised solution)**.** Let u be a solution to (1.4), which is almost periodic modulo symmetries with parameters $N(t), x(t), \xi(t)$. We say that u is *normalised* if the lifespan I contains zero and

$$N(0) = 1, \quad x(0) = \xi(0) = 0.$$

More generally, we can define the *normalisation* of a solution u at a time $t_0 \in I$ by

$$u^{[t_0]} := T_{g_{0,-\xi(t_0)/N(t_0),-x(t_0)N(t_0),N(t_0)}}\big(u(\cdot + t_0)\big). \tag{5.32}$$

Observe that $u^{[t_0]}$ is a normalised solution which is almost periodic modulo symmetries and has lifespan

$$I^{[t_0]} := \{s \in \mathbb{R} : t_0 + sN(t_0)^{-2} \in I\}$$

(so, in particular, $0 \in I^{[t_0]}$). The parameters of $u^{[t_0]}$ are given by

$$\begin{aligned} N^{[t_0]}(s) &:= \frac{N\big(t_0 + sN(t_0)^{-2}\big)}{N(t_0)} \\ \xi^{[t_0]}(s) &:= \frac{\xi\big(t_0 + sN(t_0)^{-2}\big) - \xi(t_0)}{N(t_0)} \\ x^{[t_0]}(s) &:= N(t_0)\big[x\big(t_0 + sN(t_0)^{-2}\big) - x(t_0)\big] - 2\frac{\xi(t_0)}{N(t_0)}s \end{aligned} \tag{5.33}$$

and it has the same compactness modulus function as u. Furthermore, if u is a maximal-lifespan solution then so is $u^{[t_0]}$. A similar definition can be made in the energy-critical case.

Lemma 5.17 (Compactness of normalized almost periodic solutions)**.** *Let u_n be a sequence of normalised maximal-lifespan solutions to* (1.4) *with lifespans $I_n \ni 0$, which are almost periodic modulo symmetries with parameters N_n, x_n, ξ_n and a uniform compactness modulus function C. Assume that we also have a uniform mass bound*

$$0 < \inf_n M(u_n) \le \sup_n M(u_n) < \infty. \tag{5.34}$$

Then, after passing to a subsequence if necessary, there exists a non-zero maximal-lifespan solution u to (1.4) *with lifespan $I \ni 0$ that is almost periodic modulo symmetries, such that u_n converge locally uniformly to u. A similar statement holds in the energy-critical setting.*

Proof. By hypothesis and Definition 5.1, we see that for every $\varepsilon > 0$ there exists $R > 0$ such that

$$\int_{|x|\ge R} |u_n(0,x)|^2 \, dx \le \varepsilon$$

and

$$\int_{|\xi|\ge R} |\widehat{u_n}(0,\xi)|^2 \, d\xi \le \varepsilon$$

for all n. From this, (5.34), and Proposition A.1, we see that the sequence $u_n(0)$ is precompact in the strong topology of $L^2_x(\mathbb{R}^d)$. Thus, by passing to a subsequence if necessary, we can find $u_0 \in L^2_x(\mathbb{R}^d)$ such that $u_n(0)$ converge strongly to u_0 in $L^2_x(\mathbb{R}^d)$. From (5.34) we see that u_0 is not identically zero.

Now let u be the maximal Cauchy development of u_0 from time 0, with lifespan I. By Theorem 3.7, u_n converge locally uniformly to u. The remaining claims now follow from Lemma 5.15. □

Lemma 5.18 (Local constancy of $N(t), x(t), \xi(t)$)**.** *Let u be a non-zero maximal-lifespan solution to* (1.4) *with lifespan I that is almost periodic modulo symmetries with parameters $N(t), x(t), \xi(t)$. Then there exists a small number δ, depending on u, such that for every $t_0 \in I$ we have*

$$\big[t_0 - \delta N(t_0)^{-2}, t_0 + \delta N(t_0)^{-2}\big] \subset I \tag{5.35}$$

and

$$\begin{aligned} N(t) \sim_u N(t_0), \qquad |\xi(t) - \xi(t_0)| \lesssim_u N(t_0),\\ \big|x(t) - x(t_0) - 2(t-t_0)\xi(t_0)\big| \lesssim_u N(t_0)^{-1} \end{aligned} \tag{5.36}$$

whenever $|t - t_0| \le \delta N(t_0)^{-2}$. The same statement holds for the energy-critical NLS if we set $\xi(t) \equiv 0$.

Proof. Let us first establish (5.35). We argue by contradiction. Assume (5.35) fails. Then, there exist sequences $t_n \in I$ and $\delta_n \to 0$ such that $t_n + \delta_n N(t_n)^{-2} \notin I$ for all n. Define the normalisations $u^{[t_n]}$ of u at time t_n as in (5.32). Then, $u^{[t_n]}$ are maximal-lifespan normalised solutions whose lifespans $I^{[t_n]}$ contain 0 but not δ_n; they are also almost periodic modulo symmetries with parameters given by (5.33) and the same compactness modulus function C as u. Applying Lemma 5.17

(and passing to a subsequence if necessary), we conclude that $u^{[t_n]}$ converge locally uniformly to a maximal-lifespan solution v with some lifespan $J \ni 0$. By the local well-posedness theory, J is open and so contains δ_n for all sufficiently large n. This contradicts the local uniform convergence as, by hypothesis, δ_n does not belong to $I^{[t_n]}$. Hence (5.35) holds.

We now show (5.36). Again, we argue by contradiction, shrinking δ if necessary. Suppose one of the three claims in (5.36) failed no matter how small one selected δ. Then, one can find sequences $t_n, t_n' \in I$ such that $s_n := (t_n' - t_n)N(t_n)^2 \to 0$ but $N(t_n')/N(t_n)$ converge to either zero or infinity (if the first claim failed) or $|\xi(t_n') - \xi(t_n)|/N(t_n) \to \infty$ (if the second claim failed) or $|x(t_n') - x(t_n) - 2(t_n' - t_n)\xi(t_n)|N(t_n) \to \infty$ (if the third claim failed). If we define $u^{[t_n]}$ as before and apply Lemma 5.17 (passing to a subsequence if necessary), we see once again that $u^{[t_n]}$ converge locally uniformly to a maximal-lifespan solution v with some open lifespan $J \ni 0$. But then $N^{[t_n]}(s_n)$ converge to either zero or infinity or $\xi^{[t_n]}(s_n) \to \infty$ or $x^{[t_n]}(s_n) \to \infty$ and thus, by Definition 5.1, $u^{[t_n]}(s_n)$ converge weakly to zero. On the other hand, since s_n converge to zero and $u^{[t_n]}$ are locally uniformly convergent to $v \in C^0_{t,\mathrm{loc}} L^2_x(J \times \mathbb{R}^d)$, we may conclude that $u^{[t_n]}(s_n)$ converge strongly to $v(0)$ in $L^2_x(\mathbb{R}^d)$. Thus $v(0) = 0$ and $M(u^{[t_n]})$ converge to $M(v) = 0$. But since $M(u^{(n)}) = M(u)$, we see that u vanishes identically, a contradiction. Thus (5.36) holds. □

Corollary 5.19 ($N(t)$ at blowup)**.** *Let u be a non-zero maximal-lifespan solution to* (1.4) *with lifespan I that is almost periodic modulo symmetries with frequency scale function $N : I \to \mathbb{R}^+$. If T is any finite endpoint of I, then $N(t) \gtrsim_u |T - t|^{-1/2}$; in particular, $\lim_{t\to T} N(t) = \infty$. If I is infinite or semi-infinite, then for any $t_0 \in I$ we have $N(t) \gtrsim_u \min\{N(t_0), |t - t_0|^{-1/2}\}$. The identical statement holds for the energy-critical NLS.*

Proof. This is immediate from (5.35). □

Lemma 5.20 (Local quasi-boundedness of N)**.** *Let u be a non-zero solution to the mass-critical NLS with lifespan I that is almost periodic modulo symmetries with frequency scale function $N : I \to \mathbb{R}^+$. If K is any compact subset of I, then*

$$0 < \inf_{t\in K} N(t) \leq \sup_{t\in K} N(t) < \infty.$$

The same statement holds in the energy-critical setting.

Proof. We only prove the first inequality; the other follows similarly.

We argue by contradiction. Suppose that the first inequality fails. Then, there exists a sequence $t_n \in K$ such that $\lim_{n\to\infty} N(t_n) = 0$ and hence, by Definition 5.1, $u(t_n)$ converge weakly to zero. Since K is compact, we can assume t_n converge to a limit $t_0 \in K$. As $u \in C^0_t L^2_x(K \times \mathbb{R}^d)$, we see that $u(t_n)$ converge strongly to $u(t_0)$. Thus $u(t_0)$ must be zero, contradicting the hypothesis. □

Lemma 5.21 (Strichartz norms via $N(t)$)**.** *Let u be a non-zero solution to the mass-critical NLS with lifespan I that is almost periodic modulo symmetries with parameters $N(t), x(t), \xi(t)$. If J is any subinterval of I, then*

$$\int_J N(t)^2\,dt \lesssim_u \int_J \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+2)}{d}}\,dx\,dt \lesssim_u 1 + \int_J N(t)^2\,dt. \tag{5.37}$$

Similarly, if u is a non-zero solution to the energy-critical NLS on $I \times \mathbb{R}^d$ that is almost periodic modulo symmetries with parameters $N(t), x(t)$, then

$$\int_J N(t)^2\,dt \lesssim_u \int_J \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+2)}{d-2}}\,dx\,dt \lesssim_u 1 + \int_J N(t)^2\,dt$$

for any subinterval $J \subset I$.

Proof. We consider the mass-critical case; the claim in the energy-critical case can be proved similarly. Let u be a solution to (1.4) as in the statement of the lemma. We first prove

$$\int_J \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+2)}{d}}\,dx\,dt \lesssim_u 1 + \int_J N(t)^2\,dt. \tag{5.38}$$

Let $0 < \eta < 1$ be a small parameter to be chosen momentarily and partition J into subintervals I_j so that

$$\int_{I_j} N(t)^2\,dt \le \eta; \tag{5.39}$$

this requires at most $\eta^{-1} \times \mathrm{RHS}(5.38)$ many intervals.

For each j, we may choose $t_j \in I_j$ so that

$$N(t_j)^2 |I_j| \le 2\eta. \tag{5.40}$$

By Strichartz inequality followed by Hölder and Bernstein, we obtain

$$\begin{aligned}
\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}} &\lesssim \|e^{i(t-t_j)\Delta} u(t_j)\|_{L_{t,x}^{\frac{2(d+2)}{d}}} + \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{d+4}{d}} \\
&\lesssim \|u_{\ge N_0}(t_j)\|_{L_x^2} + \|e^{i(t-t_j)\Delta} u_{\le N_0}(t_j)\|_{L_{t,x}^{\frac{2(d+2)}{d}}} + \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{d+4}{d}} \\
&\lesssim \|u_{\ge N_0}(t_j)\|_{L_x^2} + |I_j|^{\frac{d}{2(d+2)}} N_0^{\frac{d}{d+2}} \|u(t_j)\|_{L_x^2} + \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{d+4}{d}},
\end{aligned}$$

where all spacetime norms are taken on the slab $I_j \times \mathbb{R}^d$. Choosing N_0 as a large multiple of $N(t_j)$ and using Definition 5.1, one can make the first term as small as one wishes. Subsequently, choosing η sufficiently small depending on $M(u)$ and invoking (5.40), one may also render the second term arbitrarily small. Thus, by the usual bootstrap argument we obtain

$$\int_{I_j} \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+2)}{d}}\,dx\,dt \le 1.$$

Using the bound on the number of intervals I_j, this leads to (5.38).

Now we prove

$$\int_J \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+2)}{d}}\,dx\,dt \gtrsim_u \int_J N(t)^2\,dt. \tag{5.41}$$

Using Definition 5.1 and choosing η sufficiently small depending on $M(u)$, we can guarantee that

$$\int_{|x-x(t)| \le C(\eta) N(t)^{-1}} |u(t,x)|^2\,dx \gtrsim_u 1 \tag{5.42}$$

for all $t \in J$. On the other hand, a simple application of Hölder's inequality yields

$$\int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+2)}{d}}\,dx \gtrsim_u \Big(\int_{|x-x(t)|\leq C(\eta)N(t)^{-1}} |u(t,x)|^2\Big)^{\frac{d+2}{d}} N(t)^2.$$

Thus, using (5.42) and integrating over J we derive (5.41). □

Corollary 5.22 (Maximal-lifespan almost periodic solutions blow up)**.** *Let u be a maximal-lifespan solution to the mass- or energy-critical NLS that is almost periodic modulo symmetries. Then u blows up both forward and backward in time.*

Proof. In the case of a finite endpoint, this amounts to the definition of maximal-lifespan; see Corollary 3.5. Indeed, the assumption of almost-periodicity is redundant in this case.

In the case of an infinite endpoint, we see that by Corollary 5.19, $N(t) \gtrsim_u \langle t-t_0\rangle^{-1/2}$. Thus by Lemma 5.21, the spacetime norm diverges, which is the definition of blowup. □

We end this subsection with a result concerning the behaviour of almost periodic solutions at the endpoints of their maximal lifespan.

Proposition 5.23 (Asymptotic orthogonality to free evolutions, [**96**])**.** *Let $u : I\times\mathbb{R}^d \to \mathbb{C}$ be a maximal-lifespan solution to* (1.4) *that is almost periodic modulo symmetries. Then $e^{-it\Delta}u(t)$ converges weakly to zero in $L^2_x(\mathbb{R}^d)$ as $t\to\sup I$ or $t\to\inf I$. In particular, we have the 'reduced' Duhamel formulae*

$$\begin{aligned} u(t) &= i \lim_{T\to\sup I}\int_t^T e^{i(t-t')\Delta}F(u(t'))\,dt' \\ &= -i \lim_{T\to\inf I}\int_T^t e^{i(t-t')\Delta}F(u(t'))\,dt', \end{aligned} \tag{5.43}$$

where the limits are to be understood in the weak L^2_x topology. In the energy-critical case, the same formulae hold in the weak $\dot H^1_x$ topology.

Proof. Let us just prove the claim as $t\to\sup I$, since the reverse claim is similar.

Assume first that $\sup I<\infty$. Then by Corollary 5.19,

$$\lim_{t\to\sup I} N(t)=\infty.$$

By Definition 5.1, this implies that $u(t)$ converges weakly to zero as $t\to\sup I$. As $\sup I<\infty$ and the map $t\mapsto e^{it\Delta}$ is continuous in the strong operator topology on L^2_x, we see that $e^{-it\Delta}u(t)$ converges weakly to zero, as desired.

Now suppose instead that $\sup I=\infty$. It suffices to show that

$$\lim_{t\to\infty}\big\langle u(t), e^{it\Delta}\phi\big\rangle_{L^2_x(\mathbb{R}^d)}=0$$

for all test functions $\phi\in C^\infty_c(\mathbb{R}^d)$. Let $\eta>0$ be a small parameter; using Hölder's inequality and Definition 5.1, we estimate

$$\begin{aligned} &\Big|\big\langle u(t), e^{it\Delta}\phi\big\rangle_{L^2_x(\mathbb{R}^d)}\Big|^2 \\ &\lesssim \Big|\int_{|x-x(t)|\leq C(\eta)/N(t)} u(t,x)\overline{e^{it\Delta}\phi(x)}\,dx\Big|^2 + \Big|\int_{|x-x(t)|\geq C(\eta)/N(t)} u(t,x)\overline{e^{it\Delta}\phi(x)}\,dx\Big|^2 \end{aligned}$$

$$\lesssim \int_{|x-x(t)|\leq C(\eta)/N(t)} |e^{it\Delta}\phi(x)|^2\,dx + \eta\|\phi\|_{L_x^2}^2.$$

The claim now follows from Lemma 4.12, Corollary 5.19, and an easy change of variables. □

5.4. Further refinements: the enemies. The purpose of this subsection is to construct more refined counterexamples than those provided by Theorems 5.2 and 5.12, should the global well-posedness and scattering conjectures fail. These theorems provide little information about the behaviour of $N(t)$ over the lifespan I of the solution. In this subsection we strengthen those results by showing that the failure of Conjecture 1.4 or 1.5 implies the existence of at least one of three types of almost periodic solutions u for which $N(t)$ and I have very particular properties.

We would like to point out that elementary scaling arguments show that one may assume that $N(t)$ is either bounded from above or from below at least on half of its maximal lifespan; see for example, [**97**, Theorem 3.3] or [**38, 57**]. However, several recent results seem to require finer control on the nature of the blowup as one approaches either endpoint of the interval I.

We start with the mass-critical equation.

Theorem 5.24 (Three enemies: the mass-critical NLS, [**43**]). *Fix μ, d and suppose that Conjecture 1.4 fails for this choice of μ and d. Then there exists a maximal-lifespan solution u to* (1.4)*, which is almost periodic modulo symmetries, blows up both forward and backward in time, and in the focusing case also obeys $M(u) < M(Q)$.*

We can also ensure that the lifespan I and the frequency scale function $N(t)$ match one of the following three scenarios:

I. *(Soliton-like solution) We have $I = \mathbb{R}$ and*

$$N(t) = 1 \quad \textit{for all} \quad t \in \mathbb{R}.$$

II. *(Double high-to-low frequency cascade) We have $I = \mathbb{R}$,*

$$\liminf_{t\to-\infty} N(t) = \liminf_{t\to+\infty} N(t) = 0, \quad \textit{and} \quad \sup_{t\in\mathbb{R}} N(t) < \infty.$$

III. *(Self-similar solution) We have $I = (0, +\infty)$ and*

$$N(t) = t^{-1/2} \quad \textit{for all} \quad t \in I.$$

Proof. Fix μ and d. Invoking Theorem 5.2, we can find a solution v with maximal lifespan J, which is almost periodic modulo symmetries and blows up both forward and backward in time; also, in the focusing case we have $M(v) < M(Q)$.

Let $N_v(t)$ be the frequency scale function associated to v as in Definition 5.1, and let $C : \mathbb{R}^+ \to \mathbb{R}^+$ be its compactness modulus function. The solution v partially satisfies the conclusions of Theorem 5.24, but we are not necessarily in one of the three scenarios listed there. To extract a solution u with these additional properties, we will have to perform some further manipulations primarily based on the scaling and time-translation symmetries.

For any $T \geq 0$, define the quantity

$$\operatorname{osc}(T) := \inf_{t_0\in J} \frac{\sup\{N_v(t) : t \in J \text{ and } |t-t_0| \leq TN_v(t_0)^{-2}\}}{\inf\{N_v(t) : t \in J \text{ and } |t-t_0| \leq TN_v(t_0)^{-2}\}}. \tag{5.44}$$

Roughly speaking, this measures the least possible oscillation one can find in N_v on time intervals of normalised duration T. This quantity is clearly non-decreasing in T. If $\operatorname{osc}(T)$ is bounded, we will be able to extract a soliton-like solution; this is

Case I: $\lim_{T\to\infty} \operatorname{osc}(T) < \infty$.

In this case, we have arbitrarily long periods of stability for N_v. More precisely, we can find a finite number $A = A_v$, a sequence t_n of times in J, and a sequence $T_n \to \infty$ such that

$$\frac{\sup\{N_v(t) : t \in J \text{ and } |t - t_n| \leq T_n N_v(t_n)^{-2}\}}{\inf\{N_v(t) : t \in J \text{ and } |t - t_n| \leq T_n N_v(t_n)^{-2}\}} < A$$

for all n. Note that this, together with Lemma 5.18, implies that

$$[t_n - T_n N_v(t_n)^{-2}, t_n + T_n N_v(t_n)^{-2}] \subset J$$

and

$$N_v(t) \sim_v N_v(t_n)$$

for all t in this interval.

Now define the normalisations $v^{[t_n]}$ of v at times t_n as in (5.32). Then $v^{[t_n]}$ is a maximal-lifespan normalised solution with lifespan

$$J_n := \{s \in \mathbb{R} : t_n + N_v(t_n)^{-2} s \in J\} \supset [-T_n, T_n]$$

and mass $M(v)$. It is almost periodic modulo scaling with frequency scale function

$$N_{v^{[t_n]}}(s) := \frac{N_v\big(t_n + N_v(t_n)^{-2} s\big)}{N_v(t_n)}$$

and compactness modulus function C. In particular, we see that

$$N_{v^{[t_n]}}(s) \sim_v 1 \tag{5.45}$$

for all $s \in [-T_n, T_n]$.

We now apply Lemma 5.17 and conclude (passing to a subsequence if necessary) that $v^{[t_n]}$ converge locally uniformly to a maximal-lifespan solution u with mass $M(v)$ defined on an open interval I containing 0 and which is almost periodic modulo symmetries. As $T_n \to \infty$, Lemma 5.15 and (5.45) imply that the frequency scale function $N : I \to \mathbb{R}^+$ of u satisfies

$$0 < \inf_{t\in I} N(t) \leq \sup_{t\in I} N(t) < \infty.$$

In particular, by Corollary 5.19, $I = \mathbb{R}$. By modifying C by a bounded factor we may now normalise $N \equiv 1$. We have thus constructed a soliton-like solution in the sense of Theorem 5.24.

When $\operatorname{osc}(T)$ is unbounded, we must seek a solution belonging to one of the remaining two scenarios. To distinguish between them, we introduce the quantity

$$a(t_0) := \frac{\inf_{t\in J: t\leq t_0} N_v(t) + \inf_{t\in J: t\geq t_0} N_v(t)}{N_v(t_0)}$$

for every $t_0 \in J$. This measures the extent to which $N_v(t)$ decays to zero on both sides of t_0. Clearly, this quantity takes values in the interval $[0, 2]$.

Case II: $\lim_{T\to\infty} \operatorname{osc}(T) = \infty$ and $\inf_{t_0\in J} a(t_0) = 0$.

In this case, there are no long periods of stability but there are times about which there are arbitrarily large cascades from high to low frequencies in both future and past directions. This will allow us to extract a solution with a double high-to-low frequency cascade as defined in Theorem 5.24.

As $\inf_{t_0 \in J} a(t_0) = 0$, there exists a sequence of times $t_n \in J$ such that $a(t_n) \to 0$ as $n \to \infty$. By the definition of a, we can also find times $t_n^- < t_n < t_n^+$ with $t_n^-, t_n^+ \in J$ such that

$$\frac{N_v(t_n^-)}{N_v(t_n)} \to 0 \quad \text{and} \quad \frac{N_v(t_n^+)}{N_v(t_n)} \to 0.$$

Choose $t_n^- < t'_n < t_n^+$ so that

$$N_v(t'_n) \sim \sup_{t_n^- \le t \le t_n^+} N_v(t);$$

then,

$$\frac{N_v(t_n^-)}{N_v(t'_n)} \to 0 \quad \text{and} \quad \frac{N_v(t_n^+)}{N_v(t'_n)} \to 0.$$

We define the rescaled and translated times $s_n^- < 0 < s_n^+$ by

$$s_n^\pm := N_v(t'_n)^2 (t_n^\pm - t'_n)$$

and the normalisations $v^{[t'_n]}$ at times t'_n by (5.32). These are normalised maximal-lifespan solutions with lifespans containing $[s_n^-, s_n^+]$, which are almost periodic modulo G with frequency scale functions

$$N_{v^{[t'_n]}}(s) := \frac{N_v\big(t'_n + N_v(t'_n)^{-2} s\big)}{N_v(t'_n)}. \tag{5.46}$$

By the way we chose t'_n, we see that

$$N_{v^{[t'_n]}}(s) \lesssim 1 \tag{5.47}$$

for all $s_n^- \le s \le s_n^+$. Moreover,

$$N_{v^{[t'_n]}}(s_n^\pm) \to 0 \quad \text{as} \quad n \to \infty \tag{5.48}$$

for either choice of sign.

We now apply Lemma 5.17 and conclude (passing to a subsequence if necessary) that $v^{[t'_n]}$ converge locally uniformly to a maximal-lifespan solution u of mass $M(v)$ defined on an open interval I containing 0, which is almost periodic modulo symmetries.

Let N be a frequency scale function for u. From Lemma 5.20 we see that $N(t)$ is bounded from below on any compact set $K \subset I$. From this and Lemma 5.15 (and Lemma 5.13), we see that $N_{v^{[t'_n]}}(t)$ is also bounded from below, uniformly in $t \in K$, for all sufficiently large n (depending on K). As a consequence of this and (5.48), we see that s_n^- and s_n^+ cannot have any limit points in K; thus $K \subset [s_n^-, s_n^+]$ for all sufficiently large n. Therefore, $s_n^\pm$ converge to the endpoints of I. Combining this with Lemma 5.15 and (5.47), we conclude that

$$\sup_{t \in I} N(t) < \infty. \tag{5.49}$$

Corollary 5.19 now implies that I has no finite endpoints, that is, $I = \mathbb{R}$.

In order to prove that u is a double high-to-low frequency cascade, we merely need to show that

$$\liminf_{t \to +\infty} N(t) = \liminf_{t \to -\infty} N(t) = 0. \tag{5.50}$$

By time reversal symmetry, it suffices to establish that $\liminf_{t\to+\infty} N(t) = 0$. Suppose that this is not the case. Then, using (5.49) we may deduce

$$N(t) \sim_u 1$$

for all $t \geq 0$. We conclude from Lemma 5.15 that for every $m \geq 1$, there exists an n_m such that

$$N_{v^{[t'_{n_m}]}}(t) \sim_u 1$$

for all $0 \leq t \leq m$. But by (5.44) and (5.46) this implies that

$$\operatorname{osc}(\varepsilon m) \lesssim_u 1$$

for all m and some $\varepsilon = \varepsilon(u) > 0$ independent of m. Note that ε is chosen as a lower bound on the quantities $N(t''_{n_m})^2/N(t'_{n_m})^2$ where $t''_{n_m} = t'_{n_m} + \frac{m}{2}N(t'_{n_m})^{-2}$. This contradicts the hypothesis $\lim_{T\to\infty} \operatorname{osc}(T) = \infty$ and so settles Case II.

Case III: $\lim_{T\to\infty} \operatorname{osc}(T) = \infty$ and $\inf_{t_0\in J} a(t_0) > 0$.

In this case, there are no long periods of stability and no double cascades from high to low frequencies; we will be able to extract a self-similar solution in the sense of Theorem 5.24.

Let $\varepsilon = \varepsilon(v) > 0$ be such that $\inf_{t_0\in J} a(t_0) \geq 2\varepsilon$. We call a time t_0 *future-focusing* if

$$N_v(t) \geq \varepsilon N_v(t_0) \text{ for all } t \in J \text{ with } t \geq t_0 \tag{5.51}$$

and *past-focusing* if

$$N_v(t) \geq \varepsilon N_v(t_0) \text{ for all } t \in J \text{ with } t \leq t_0. \tag{5.52}$$

From the choice of ε we see that every time $t_0 \in J$ is either future-focusing or past-focusing, or possibly both.

We will now show that either all sufficiently late times are future-focusing or that all sufficiently early times are past-focusing. If this were false, there would be a future-focusing time t_0 and a sequence of past-focusing times t_n that converge to $\sup J$. For sufficiently large n, we have $t_n \geq t_0$. By (5.51) and (5.52) we then see that

$$N_v(t_n) \sim_v N_v(t_0)$$

for all such n. For any $t_0 < t < t_n$, we know that t is either past-focusing or future-focusing; thus we have either $N_v(t_0) \geq \varepsilon N_v(t)$ or $N_v(t_n) \geq \varepsilon N_v(t)$. Also, since t_0 is future-focusing, $N_v(t) \geq \varepsilon N_v(t_0)$. We conclude that

$$N_v(t) \sim_v N_v(t_0)$$

for all $t_0 < t < t_n$; since $t_n \to \sup J$, this claim in fact holds for all $t_0 < t < \sup J$. In particular, from Corollary 5.19 we see that v does not blow up forward in finite time, that is, $\sup J = \infty$. The function N_v is now bounded above and below on the interval $(t_0, +\infty)$, which implies that $\lim_{T\to\infty} \operatorname{osc}(T) < \infty$, a contradiction. This proves the assertion at the beginning of the paragraph.

We may now assume that future-focusing occurs for all sufficiently late times; more precisely, we can find $t_0 \in J$ such that all times $t \geq t_0$ are future-focusing. The case when all sufficiently early times are past-focusing reduces to this via time-reversal symmetry.

We will now recursively construct a new sequence of times t_n. More precisely, we will explain how to choose t_{n+1} from t_n.

As $\lim_{T\to\infty} \operatorname{osc}(T) = \infty$, we have $\operatorname{osc}(B) \geq 2/\varepsilon$ for some sufficiently large $B = B(v) > 0$. Given $J \ni t_n > t_0$ set $A = 2B\varepsilon^{-2}$ and $t_n' = t_n + \frac{1}{2}AN_v(t_n)^{-2}$. As $t_n > t_0$, it is future-focusing and so $N_v(t_n') \geq \varepsilon N_v(t_n)$. From this, we see that

$$\left\{t : |t - t_n'| \leq BN_v(t_n')^{-2}\right\} \subseteq \left[t_n, t_n + AN_v(t_n)^{-2}\right]$$

and thus, by the definition of B and the fact that all $t \geq t_n$ are future-focusing,

$$\sup_{t\in J\cap[t_n,t_n+AN_v(t_n)^{-2}]} N_v(t) \geq 2N_v(t_n). \tag{5.53}$$

Using this and Lemma 5.18, we see that for every $t_n \in J$ with $t_n \geq t_0$ there exists a time $t_{n+1} \in J$ obeying

$$t_n < t_{n+1} \leq t_n + AN(t_n)^{-2} \tag{5.54}$$

such that

$$2N_v(t_n) \leq N_v(t_{n+1}) \lesssim_v N_v(t_n) \tag{5.55}$$

and

$$N_v(t) \sim_v N_v(t_n) \quad \text{for all } t_n \leq t \leq t_{n+1}. \tag{5.56}$$

From (5.55) we have

$$N_v(t_n) \geq 2^n N_v(t_0)$$

for all $n \geq 0$, which by (5.54) implies

$$t_{n+1} \leq t_n + O_v(2^{-2n} N_v(t_0)^{-2}).$$

Thus t_n converge to a limit and $N_v(t_n)$ to infinity. In view of Lemma 5.20, this implies that $\sup J$ is finite and $\lim_{n\to\infty} t_n = \sup J$.

Let $n \geq 0$. By (5.55),

$$N_v(t_{n+m}) \geq 2^m N_v(t_n)$$

for all $m \geq 0$ and so, using (5.54) we obtain

$$0 < t_{n+m+1} - t_{n+m} \lesssim_v 2^{-2m} N_v(t_n)^{-2}.$$

Summing this series in m, we conclude that

$$\sup J - t_n \lesssim_v N_v(t_n)^{-2}.$$

Combining this with Corollary 5.19, we obtain

$$\sup J - t_n \sim_v N_v(t_n)^{-2}.$$

In particular, we have

$$\sup J - t_{n+1} \sim_v \sup J - t_n \sim_v N_v(t_n)^{-2}.$$

Applying (5.55) and (5.56) shows

$$\sup J - t \sim_v N_v(t)^{-2}$$

for all $t_n \leq t \leq t_{n+1}$. Since t_n converge to $\sup J$, we conclude that

$$\sup J - t \sim_v N_v(t)^{-2}$$

for all $t_0 \leq t < \sup J$.

As we have the freedom to modify $N(t)$ by a bounded function (modifying C appropriately), we may normalise

$$N_v(t) = (\sup J - t)^{-1/2}$$

for all $t_0 \le t < \sup J$. It is now not difficult to extract our sought-after self-similar solution by suitably rescaling the interval $(t_0, \sup J)$ as follows.

Consider the normalisations $v^{[t_n]}$ of v at times t_n (cf. (5.32)). These are maximal-lifespan normalised solutions of mass $M(v)$, whose lifespans include the interval

$$\Big(-\frac{\sup J - t_0}{\sup J - t_n}, 1\Big),$$

and which are almost periodic modulo scaling with compactness modulus function C and frequency scale functions

$$N_{v^{[t_n]}}(s) = (1-s)^{-1/2} \tag{5.57}$$

for all $-\frac{\sup J - t_0}{\sup J - t_n} < s < 1$. We now apply Lemma 5.17 and conclude (passing to a subsequence if necessary) that $v^{[t_n]}$ converge locally uniformly to a maximal-lifespan solution u of mass $M(v)$ defined on an open interval I containing $(-\infty, 1)$, which is almost periodic modulo symmetries.

By Lemma 5.15 and (5.57), we see that u has a frequency scale function N obeying

$$N(s) \sim_v (1-s)^{-1/2}$$

for all $s \in (-\infty, 1)$. By modifying N (and C) by a bounded factor, we may normalise

$$N(s) = (1-s)^{-1/2}.$$

From this, Lemma 5.18, and Corollary 5.19 we see that we must have $I = (-\infty, 1)$. Applying a time translation (by -1) followed by a time reversal, we obtain our sought-after self-similar solution.

This finishes the proof of Theorem 5.24. □

Finally, we identify the enemies in the energy-critical setting. The precise statement we present is not as ambitious as the one for the mass-critical NLS, but it has proven sufficient to resolve the global well-posedness and scattering conjecture in high dimensions.

Theorem 5.25 (Three enemies: the energy-critical NLS, [**44**]). *Fix μ and $d \ge 3$ and suppose that Conjecture 1.5 fails for this choice of μ and d. Then there exists a minimal kinetic energy, maximal-lifespan solution u to* (1.6)*, which is almost periodic modulo symmetries, $\|u\|_{L_{t,x}^{2(d+2)/(d-2)}(I\times\mathbb{R}^d)} = \infty$, and in the focusing case also obeys $\sup_{t\in I} \|\nabla u(t)\|_2 < \|\nabla W\|_2$.*

We can also ensure that the lifespan I and the frequency scale function $N : I \to \mathbb{R}^+$ match one of the following three scenarios:

I. *(Finite-time blowup) We have that either $|\inf I| < \infty$ or $\sup I < \infty$.*

II. *(Soliton-like solution) We have $I = \mathbb{R}$ and*

$$N(t) = 1 \quad \textit{for all} \quad t \in \mathbb{R}.$$

III. *(Low-to-high frequency cascade) We have $I = \mathbb{R}$,*

$$\inf_{t\in\mathbb{R}} N(t) \ge 1, \quad \textit{and} \quad \limsup_{t\to+\infty} N(t) = \infty.$$

Proof. Exercise: adapt the proof of Theorem 5.24 to cover this case. □

6. Quantifying the compactness

In this section we continue our study of minimal blowup solutions, particularly, the study of the enemies described in Theorems 5.24 and 5.25. As we have seen in Section 5, one of properties that these minimal blowup solutions enjoy is that their orbit is precompact (modulo symmetries) in L_x^2 (in the mass-critical case) or in $\dot{H}_x^1$ (in the energy-critical case). We will now show that these minimal counterexamples to the global well-posedness and scattering conjectures enjoy additional regularity and decay, properties which one should regard as a strengthening of the precompactness of their profiles, indeed, as a way to quantify this (pre)compactness.

The *goal* is to show that solutions corresponding to the three scenarios described in Theorem 5.24 belong to $L_t^\infty H_x^1$ (or even $L_t^\infty H_x^{1+\varepsilon}$ for some $\varepsilon = \varepsilon(d) > 0$) throughout their lifespan, while solutions corresponding to the three scenarios described in Theorem 5.25 belong to $L_t^\infty L_x^2$ (or even $L_t^\infty \dot{H}_x^{-\varepsilon}$ for some $\varepsilon = \varepsilon(d) > 0$). As we will see in Section 8, this additional regularity and decay is sufficient to preclude the enemies to the global well-posedness and scattering conjectures. To give just a quick example of how this works, let us notice that in order to preclude the self-similar solution described in Theorem 5.24, it suffices to prove that such a solution belongs to $L_t^\infty H_x^1$, since then it is global (see Weinstein [**105**] for the focusing case); this contradicts the fact that a self-similar solution blows up at $t = 0$.

The goal described in the paragraph above is by no means easily achievable; indeed, most of the effort and innovation in proving the global well-posedness and scattering conjectures concentrate in attaining this goal. In the mass-critical case, additional regularity for the enemies described in Theorem 5.24 was so far only proved in dimensions $d \geq 2$ under the additional assumption of spherical symmetry on the initial data; see [**43, 46**] and also [**97**]. Removing the spherical symmetry assumption even in the defocusing case (when one has the advantage of using Morawetz-type inequalities) has proven quite difficult and is still an open problem.

In the energy-critical case, the goal was achieved in dimensions $d \geq 5$ in [**44**], thus resolving the global well-posedness and scattering conjecture in this case. In lower dimensions $d = 3, 4$, the conjecture was only proved under the additional assumption of spherical symmetry on the initial data; see [**38**]. Unlike in the mass-critical case, for the energy-critical NLS this assumption is sufficiently strong that one does not need to achieve the goal in order to rule out the enemies. Indeed, in these low dimensions, the goal described above is presumably too ambitious since even the ground state W does not belong to L_x^2 in this case. Removing the spherical symmetry assumption for $d = 3, 4$ remains quite a challenge.

In the mass-critical case, we will only revisit the proof of additional regularity for the self-similar solution (cf. Theorem 5.24) and only in the spherically symmetric case, as it appears in [**43, 46**]. We will, however, present the complete argument for the energy-critical NLS in dimensions $d \geq 5$, following [**44**].

6.1. Additional regularity: the self-similar scenario.

Theorem 6.1 (Regularity in the self-similar case, [**43, 46**]). *Let $d \geq 2$ and let u be a spherically symmetric solution to* (1.4) *that is almost periodic modulo scaling and self-similar in the sense of Theorem 5.24. Then $u(t) \in H_x^s(\mathbb{R}^d)$ for all $t \in (0, \infty)$ and all $0 \leq s < 1 + \frac{4}{d}$.*

Corollary 6.2 (Absence of self-similar solutions). *For $d \geq 2$ there are no spherically symmetric solutions to* (1.4) *that are self-similar in the sense of Theorem 5.24.*

PROOF. By Theorem 6.1, any such solution would obey $u(t) \in H^1_x(\mathbb{R}^d)$ for all $t \in (0,\infty)$. Then, by the H^1_x global well-posedness theory (see Corollary 4.3 in the focusing case), there exists a global solution with initial data $u(t_0)$ at some time $t_0 \in (0,\infty)$. On the other hand, self-similar solutions blow up at time $t = 0$. These two facts (combined with the uniqueness statement in Corollary 3.5) yield a contradiction. □

The remainder of this subsection is devoted to proving Theorem 6.1.

Let u be as in Theorem 6.1. For any $A > 0$, we define

$$\begin{aligned} \mathcal{M}(A) &:= \sup_{T>0} \|u_{>AT^{-1/2}}(T)\|_{L^2_x(\mathbb{R}^d)} \\ \mathcal{S}(A) &:= \sup_{T>0} \|u_{>AT^{-1/2}}\|_{L^{2(d+2)/d}_{t,x}([T,2T]\times\mathbb{R}^d)} \\ \mathcal{N}(A) &:= \sup_{T>0} \|P_{>AT^{-1/2}} F(u)\|_{L^{2(d+2)/(d+4)}_{t,x}([T,2T]\times\mathbb{R}^d)}. \end{aligned} \tag{6.1}$$

The notation chosen indicates the quantity being measured, namely, the mass, the symmetric Strichartz norm, and the nonlinearity in the adjoint Strichartz norm, respectively. As u is self-similar, $N(t)$ is comparable to $T^{-1/2}$ for t in the interval $[T,2T]$. Thus, the Littlewood-Paley projections are adapted to the natural frequency scale on each dyadic time interval.

To prove Theorem 6.1 it suffices to show that for every $0 < s < 1 + \frac{4}{d}$ we have

$$\mathcal{M}(A) \lesssim_{s,u} A^{-s}, \tag{6.2}$$

whenever A is sufficiently large depending on u and s. To establish this, we need a variety of estimates linking $\mathcal{M}$, $\mathcal{S}$, and $\mathcal{N}$. From mass conservation, Lemma 5.21, self-similarity, and Hölder's inequality, we see that

$$\mathcal{M}(A) + \mathcal{S}(A) + \mathcal{N}(A) \lesssim_u 1 \tag{6.3}$$

for all $A > 0$. From the Strichartz inequality (Theorem 3.2), we also see that

$$\mathcal{S}(A) \lesssim \mathcal{M}(A) + \mathcal{N}(A) \tag{6.4}$$

for all $A > 0$. One more application of Strichartz inequality combined with Lemma 5.21 and self-similarity shows

$$\|u\|_{L^2_t L^{\frac{2d}{d-2}}_x([T,2T]\times\mathbb{R}^d)} \lesssim_u 1. \tag{6.5}$$

Next, we obtain a deeper connection between these quantities.

Lemma 6.3 (Nonlinear estimate)**.** *Let $\eta > 0$ and $0 < s < 1 + \frac{4}{d}$. For all $A > 100$ and $0 < \beta \le 1$, we have*

$$\begin{aligned} \mathcal{N}(A) \lesssim_u \sum_{N \le \eta A^\beta} \big(\tfrac{N}{A}\big)^s \mathcal{S}(N) &+ \big[\mathcal{S}(\eta A^{\frac{\beta}{2(d-1)}}) + \mathcal{S}(\eta A^\beta)\big]^{\frac{4}{d}} \mathcal{S}(\eta A^\beta) \\ &+ A^{-\frac{2\beta}{d^2}} \big[\mathcal{M}(\eta A^\beta) + \mathcal{N}(\eta A^\beta)\big]. \end{aligned} \tag{6.6}$$

PROOF. Fix $\eta > 0$ and $0 < s < 1 + \frac{4}{d}$. It suffices to bound

$$\big\|P_{>AT^{-\frac12}} F(u)\big\|_{L^{\frac{2(d+2)}{d+4}}_{t,x}([T,2T]\times\mathbb{R}^d)}$$

by the right-hand side of (6.6) for fixed $T > 0$, $A > 100$, and $0 < \beta \le 1$.

To achieve this, we decompose

$$F(u) = F(u_{\leq \eta A^\beta T^{-\frac12}}) + O\big(|u_{\leq \eta A^\alpha T^{-\frac12}}|^{\frac4d}|u_{>\eta A^\beta T^{-\frac12}}|\big) + O\big(|u_{\eta A^\alpha T^{-\frac12}<\cdot\leq \eta A^\beta T^{-\frac12}}|^{\frac4d}|u_{>\eta A^\beta T^{-\frac12}}|\big) + O\big(|u_{>\eta A^\beta T^{-\frac12}}|^{1+\frac4d}\big), \tag{6.7}$$

where $\alpha = \frac{\beta}{2(d-1)}$. To estimate the contribution from the last two terms in the expansion above, we discard the projection onto high frequencies and then use Hölder's inequality and (6.1):

$$\big\| |u_{\eta A^\alpha T^{-\frac12}<\cdot\leq \eta A^\beta T^{-\frac12}}|^{\frac4d} u_{>\eta A^\beta T^{-\frac12}}\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([T,2T]\times\mathbb{R}^d)} \lesssim \mathcal{S}(\eta A^\alpha)^{\frac4d}\mathcal{S}(\eta A^\beta)$$

$$\big\| |u_{>\eta A^\beta T^{-\frac12}}|^{1+\frac4d}\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([T,2T]\times\mathbb{R}^d)} \lesssim \mathcal{S}(\eta A^\beta)^{1+\frac4d}.$$

To estimate the contribution coming from second term on the right-hand side of (6.7), we discard the projection onto high frequencies and then use Hölder's inequality, Lemma A.6, Corollary 4.19, and (6.4):

$$\begin{aligned}
&\big\|P_{>AT^{-\frac12}} O\big(|u_{\leq \eta A^\alpha T^{-\frac12}}|^{\frac4d}|u_{>\eta A^\beta T^{-\frac12}}|\big)\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([T,2T]\times\mathbb{R}^d)}\\
&\lesssim \big\|u_{\leq \eta A^\alpha T^{-\frac12}} u_{>\eta A^\beta T^{-\frac12}}\big\|_{L^2_{t,x}([T,2T]\times\mathbb{R}^d)}^{\frac{8}{d^2}} \big\|u_{>\eta A^\beta T^{-\frac12}}\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}([T,2T]\times\mathbb{R}^d)}^{1-\frac{8}{d^2}}\\
&\qquad\times \big\|u_{\leq \eta A^\alpha T^{-\frac12}}\big\|_{L^2_{t,x}([T,2T]\times\mathbb{R}^d)}^{\frac4d-\frac{8}{d^2}}\\
&\lesssim_u \big[(\eta A^\beta T^{-\frac12})^{-\frac12}(\eta A^\alpha T^{-\frac12})^{\frac{d-1}{2}}\big]^{\frac{8}{d^2}}\big[\mathcal{M}(\eta A^\beta)+\mathcal{N}(\eta A^\beta)\big]^{\frac{8}{d^2}}\mathcal{S}(\eta A^\beta)^{1-\frac{8}{d^2}}T^{\frac2d-\frac{4}{d^2}}\\
&\lesssim_u A^{-\frac{2\beta}{d^2}}\big[\mathcal{M}(\eta A^\beta)+\mathcal{N}(\eta A^\beta)\big].
\end{aligned}$$

We now turn to the first term on the right-hand side of (6.7). By Lemma A.6 and Corollary A.14 combined with (6.3), we estimate

$$\begin{aligned}
\|P_{>AT^{-\frac12}} F(u_{\leq \eta A^\beta T^{-\frac12}})\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([T,2T]\times\mathbb{R}^d)}
&\lesssim (AT^{-\frac12})^{-s}\big\||\nabla|^s F(u_{\leq \eta A^\beta T^{-\frac12}})\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([T,2T]\times\mathbb{R}^d)}\\
&\lesssim_u (AT^{-\frac12})^{-s}\big\||\nabla|^s u_{\leq \eta A^\beta T^{-\frac12}}\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}([T,2T]\times\mathbb{R}^d)}\\
&\lesssim_u \sum_{N\leq \eta A^\beta}\big(\tfrac{N}{A}\big)^s\mathcal{S}(N),
\end{aligned}$$

which is acceptable. This finishes the proof of the lemma. □

We have some decay as $A\to\infty$:

Lemma 6.4 (Qualitative decay)**.** *We have*

$$\lim_{A\to\infty}\mathcal{M}(A) = \lim_{A\to\infty}\mathcal{S}(A) = \lim_{A\to\infty}\mathcal{N}(A) = 0. \tag{6.8}$$

Proof. The vanishing of the first limit follows from Definition 5.1, (6.1), and self-similarity. By interpolation, (6.1), and (6.5),

$$\mathcal{S}(A) \lesssim \mathcal{M}(A)^{\frac{2}{d+2}}\|u_{\geq AT^{-\frac12}}\|_{L_t^2 L_x^{\frac{2d}{d-2}}([T,2T]\times\mathbb{R}^d)}^{\frac{d}{d+2}} \lesssim_u \mathcal{M}(A)^{\frac{2}{d+2}}.$$

Thus, as the first limit in (6.8) vanishes, we obtain that the second limit vanishes. The vanishing of the third limit follows from that of the second and Lemma 6.3. □

We have now gathered enough tools to prove some regularity, albeit in the symmetric Strichartz space. As such, the next result is the crux of this subsection.

Proposition 6.5 (Quantitative decay estimate)**.** *Let $0<\eta<1$ and $0<s<1+\frac{4}{d}$. If η is sufficiently small depending on u and s, and A is sufficiently large depending on u, s, and η,*

$$\mathcal{S}(A) \leq \sum_{N\leq \eta A} \big(\tfrac{N}{A}\big)^s \mathcal{S}(N) + A^{-\frac{1}{d^2}}. \tag{6.9}$$

In particular, by Lemma A.15,

$$\mathcal{S}(A) \lesssim_u A^{-\frac{1}{d^2}}, \tag{6.10}$$

for all $A>0$.

Proof. Fix $\eta\in(0,1)$ and $0<s<1+\frac{4}{d}$. To establish (6.9), it suffices to show

$$\big\| u_{>AT^{-1/2}}\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}([T,2T]\times\mathbb{R}^d)} \lesssim_{u,\varepsilon} \sum_{N\leq\eta A} \big(\tfrac{N}{A}\big)^{s+\varepsilon}\mathcal{S}(N) + A^{-\frac{3}{2d^2}} \tag{6.11}$$

for all $T>0$ and some small $\varepsilon=\varepsilon(d,s)>0$, since then (6.9) follows by requiring η to be small and A to be large, both depending upon u.

Fix $T>0$. By writing the Duhamel formula (3.12) beginning at $\frac{T}{2}$ and then using the Strichartz inequality, we obtain

$$\begin{aligned}\big\| u_{>AT^{-1/2}}\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}([T,2T]\times\mathbb{R}^d)} \lesssim{}& \big\| P_{>AT^{-1/2}} e^{i(t-\frac{T}{2})\Delta} u(\tfrac{T}{2})\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}([T,2T]\times\mathbb{R}^d)}\\ &+ \big\| P_{>AT^{-1/2}} F(u)\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([\frac{T}{2},2T]\times\mathbb{R}^d)}.\end{aligned}$$

Consider the second term. By (6.1), we have

$$\big\| P_{>AT^{-1/2}} F(u)\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([\frac{T}{2},2T]\times\mathbb{R}^d)} \lesssim \mathcal{N}(A/2).$$

Using Lemma 6.3 (with $\beta=1$ and s replaced by $s+\varepsilon$ for some $0<\varepsilon<1+\frac{4}{d}-s$) combined with Lemma 6.4 (choosing A sufficiently large depending on u, s, and η), and (6.3), we derive

$$\big\| P_{>AT^{-1/2}} F(u)\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}([\frac{T}{2},2T]\times\mathbb{R}^d)} \lesssim_{u,\varepsilon} \mathrm{RHS}(6.11).$$

Thus, the second term is acceptable.

We now consider the first term. It suffices to show

$$\big\| P_{>AT^{-1/2}} e^{i(t-\frac{T}{2})\Delta} u(\tfrac{T}{2})\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}([T,2T]\times\mathbb{R}^d)} \lesssim_u A^{-\frac{3}{2d^2}}, \tag{6.12}$$

which we will deduce by first proving two estimates at a single frequency scale, interpolating between them, and then summing.

From Theorem 4.29 and mass conservation, we have

$$\big\| P_{BT^{-1/2}} e^{i(t-\frac{T}{2})\Delta} u(\tfrac{T}{2})\big\|_{L^q_{t,x}([T,2T]\times\mathbb{R}^d)} \lesssim_{u,q} (BT^{-1/2})^{\frac{d}{2}-\frac{d+2}{q}} \tag{6.13}$$

for all $\frac{4d+2}{2d-1} < q \leq \frac{2(d+2)}{d}$ and $B>0$. This is our first estimate.

Using the Duhamel formula (3.12), we write

$$P_{BT^{-1/2}} e^{i(t-\frac{T}{2})\Delta} u(\tfrac{T}{2}) = P_{BT^{-1/2}} e^{i(t-\delta)\Delta} u(\delta) - i \int_\delta^{\frac{T}{2}} P_{BT^{-1/2}} e^{i(t-t')\Delta} F(u(t'))\, dt'$$

for any $\delta > 0$. By self-similarity, the former term converges strongly to zero in L_x^2 as $\delta \to 0$. Convergence to zero in $L_x^{2d/(d-2)}$ then follows from Lemma A.6. Thus, using Hölder's inequality followed by the dispersive estimate (3.2), and then (6.5), we estimate

$$\begin{aligned}
\big\| P_{BT^{-1/2}} e^{i(t-\frac{T}{2})\Delta} u(\tfrac{T}{2}) \big\|_{L_{t,x}^{\frac{2d}{d-2}}([T,2T]\times\mathbb{R}^d)}
&\lesssim T^{\frac{d-2}{2d}} \Big\| \int_0^{\frac{T}{2}} \frac{1}{t-t'} \|F(u(t'))\|_{L_x^{\frac{2d}{d+2}}}\, dt' \Big\|_{L_t^\infty([T,2T])} \\
&\lesssim T^{-\frac{d+2}{2d}} \|F(u)\|_{L_t^1 L_x^{\frac{2d}{d+2}}((0,\frac{T}{2}]\times\mathbb{R}^d)} \\
&\lesssim T^{-\frac{d+2}{2d}} \sum_{0<\tau\leq\frac{T}{4}} \|F(u)\|_{L_t^1 L_x^{\frac{2d}{d+2}}([\tau,2\tau]\times\mathbb{R}^d)} \\
&\lesssim T^{-\frac{d+2}{2d}} \sum_{0<\tau\leq\frac{T}{4}} \tau^{1/2} \|u\|_{L_t^2 L_x^{\frac{2d}{d-2}}([\tau,2\tau]\times\mathbb{R}^d)} \|u\|_{L_t^\infty L_x^2([\tau,2\tau]\times\mathbb{R}^d)}^{\frac{4}{d}} \\
&\lesssim_u T^{-1/d}.
\end{aligned}$$

Interpolating between the estimate just proved and the $q = \frac{2d(d+2)(4d-3)}{4d^3-3d^2+12}$ case of (6.13), we obtain

$$\|P_{BT^{-1/2}} e^{i(t-\frac{T}{2})\Delta} u(\tfrac{T}{2})\|_{L_{t,x}^{\frac{2(d+2)}{d}}([T,2T]\times\mathbb{R}^d)} \lesssim_u B^{-\frac{3}{2d^2}}.$$

Summing this over dyadic $B \geq A$ yields (6.12) and hence (6.11). □

Corollary 6.6. *For any $A > 0$ we have*

$$\mathcal{M}(A) + \mathcal{S}(A) + \mathcal{N}(A) \lesssim_u A^{-1/d^2}.$$

Proof. The bound on $\mathcal{S}$ was proved in the previous proposition. The bound on $\mathcal{N}$ follows from this, Lemma 6.3 with $\beta = 1$, and (6.3).

We now turn to the bound on $\mathcal{M}$. By Proposition 5.23 and weak lower semicontinuity of the norm,

$$\|P_{>AT^{-1/2}} u(T)\|_2 \leq \sum_{k=0}^{\infty} \Big\| \int_{2^k T}^{2^{k+1}T} e^{i(T-t')\Delta} P_{>AT^{-1/2}} F(u(t'))\, dt' \Big\|_2. \tag{6.14}$$

Intuitively, the reason for using the Duhamel formula forward in time is that the solution becomes smoother as $N(t) \to 0$.

Combining (6.14) with Strichartz inequality and (6.1), we get

$$\mathcal{M}(A) = \sup_{T>0} \|P_{>AT^{-1/2}} u(T)\|_2 \lesssim \sum_{k=0}^{\infty} \mathcal{N}(2^{k/2} A). \tag{6.15}$$

The desired bound on $\mathcal{M}$ now follows from that on $\mathcal{N}$. □

PROOF OF THEOREM 6.1. Let $0 < s < 1 + \frac{4}{d}$. Combining Lemma 6.3 (with $\beta = 1 - \frac{1}{2d^2}$), (6.4), and (6.15), we deduce that if

$$\mathcal{S}(A) + \mathcal{M}(A) + \mathcal{N}(A) \lesssim_u A^{-\sigma}$$

for some $0 < \sigma < s$, then

$$\mathcal{S}(A) + \mathcal{M}(A) + \mathcal{N}(A) \lesssim_u A^{-\sigma}\Big(A^{-\frac{s-\sigma}{2d^2}} + A^{-\frac{(d+1)(3d-2)\sigma}{2d^3(d-1)}} + A^{-\frac{3-\sigma}{2d^2} - \frac{d^2-2}{2d^4}}\Big).$$

More precisely, Lemma 6.3 provides the bound on $\mathcal{N}(A)$, then (6.15) gives the bound on $\mathcal{M}(A)$ and then finally (6.4) gives the bound on $\mathcal{S}(A)$.

Iterating this statement shows that $u(t) \in H^s_x(\mathbb{R}^d)$ for all $0 < s < 1 + \frac{4}{d}$. Note that Corollary 6.6 allows us to begin the iteration with $\sigma = d^{-2}$. □

6.2. Additional decay: the finite-time blowup case. We consider now the energy-critical NLS. The purpose of the next two subsections is to prove that solutions corresponding to the three scenarios described in Theorem 5.25 obey additional decay, in particular, they belong to $L_t^\infty L_x^2$ or better (at least in dimensions $d \geq 5$).

We start with the finite-time blowup scenario and show that in this case, the solution has finite mass; indeed, we will show that the solution must have zero mass, and hence derive a contradiction to the fact that it is, after all, a blowup solution. In this particular case, we do not need to restrict to dimensions $d \geq 5$. The argument is essentially taken from [**38**].

Theorem 6.7 (No finite-time blowup)**.** *Let $d \geq 3$. Then there are no maximal-lifespan solutions $u : I \times \mathbb{R}^d \to \mathbb{C}$ to* (1.6) *that are almost periodic modulo symmetries, obey*

$$S_I(u) = \infty, \tag{6.16}$$

and

$$\sup_{t\in I} \|\nabla u(t)\|_2 < \infty, \tag{6.17}$$

and are such that either $|\inf I| < \infty$ or $\sup I < \infty$.

PROOF. Suppose for a contradiction that there existed such a solution u. Without loss of generality, we may assume $\sup I < \infty$. By Corollary 5.19, we must have

$$\liminf_{t \nearrow \sup I} N(t) = \infty. \tag{6.18}$$

We now show that (6.18) implies

$$\limsup_{t\nearrow \sup I} \int_{|x|\leq R} |u(t,x)|^2\,dx = 0 \quad \text{for all } R > 0. \tag{6.19}$$

Indeed, let $0 < \eta < 1$ and $t \in I$. By Hölder's inequality, Sobolev embedding, and (6.17),

$$\begin{aligned}\int_{|x|\leq R} |u(t,x)|^2\,dx &\leq \int_{|x-x(t)|\leq \eta R} |u(t,x)|^2\,dx + \int_{\substack{|x|\leq R\\ |x-x(t)|>\eta R}} |u(t,x)|^2\,dx \\ &\lesssim \eta^2 R^2 \|u(t)\|^2_{\frac{2d}{d-2}} + R^2\Big(\int_{|x-x(t)|>\eta R} |u(t,x)|^{\frac{2d}{d-2}}\,dx\Big)^{\frac{d-2}{d}}\end{aligned}$$

$$\lesssim \eta^2 R^2 + R^2 \Big(\int_{|x-x(t)|>\eta R} |u(t,x)|^{\frac{2d}{d-2}}\, dx \Big)^{\frac{d-2}{d}}.$$

Letting $\eta \to 0$, we can make the first term on the right-hand side of the inequality above as small as we wish. On the other hand, by (6.18) and Definition 5.11, we see that

$$\limsup_{t \nearrow \sup I} \int_{|x-x(t)|>\eta R} |u(t,x)|^{\frac{2d}{d-2}}\, dx = 0.$$

This proves (6.19).

The next step is to prove that (6.19) implies the solution u is identically zero, thus contradicting (6.16). For $t \in I$ define

$$M_R(t) := \int_{\mathbb{R}^d} \phi\big(\tfrac{|x|}{R}\big) |u(x,t)|^2\, dx,$$

where ϕ is a smooth, radial function, such that

$$\phi(r) = \begin{cases} 1 & \text{for } r \leq 1 \\ 0 & \text{for } r \geq 2. \end{cases}$$

By (6.19),

$$\limsup_{t \nearrow \sup I} M_R(t) = 0 \quad \text{for all } R > 0. \tag{6.20}$$

On the other hand, a simple computation involving Hardy's inequality and (6.17) shows

$$|\partial_t M_R(t)| \lesssim \|\nabla u(t)\|_2 \Big\| \frac{u(t)}{|x|} \Big\|_2 \lesssim \|\nabla u(t)\|_2^2 \lesssim_u 1.$$

Thus, by the Fundamental Theorem of Calculus,

$$M_R(t_1) = M_R(t_2) - \int_{t_1}^{t_2} \partial_t M_R(t)\, dt \lesssim_u M_R(t_2) + |t_2 - t_1|$$

for all $t_1, t_2 \in I$ and $R > 0$. Letting $t_2 \nearrow \sup I$ and invoking (6.20), we deduce

$$M_R(t_1) \lesssim_u |\sup I - t_1|.$$

Now letting $R \to \infty$ we obtain $u(t_1) \in L^2_x(\mathbb{R}^d)$. Finally, letting $t_1 \nearrow \sup I$ and using the conservation of mass, we conclude $u \equiv 0$, contradicting (6.16).

This concludes the proof of Theorem 6.7. □

6.3. Additional decay: the global case.

In this subsection we prove

Theorem 6.8 (Negative regularity in the global case, [**44**])**.** *Let $d \geq 5$ and let u be a global solution to* (1.6) *that is almost periodic modulo symmetries. Suppose also that*

$$\sup_{t \in \mathbb{R}} \|\nabla u(t)\|_{L^2_x} < \infty \tag{6.21}$$

and

$$\inf_{t \in \mathbb{R}} N(t) \geq 1. \tag{6.22}$$

Then $u \in L^\infty_t \dot{H}^{-\varepsilon}_x(\mathbb{R} \times \mathbb{R}^d)$ for some $\varepsilon = \varepsilon(d) > 0$. In particular, $u \in L^\infty_t L^2_x$.

The proof of Theorem 6.8 is achieved in two steps: First, we 'break' scaling in a Lebesque space; more precisely, we prove that our solution lives in $L_t^\infty L_x^p$ for some $2 < p < \frac{2d}{d-2}$. Next, we use a double Duhamel trick to upgrade this to $u \in L_t^\infty \dot H_x^{1-s}$ for some $s = s(p,d) > 0$. Iterating the second step finitely many times, we derive Theorem 6.8.

The double Duhamel trick was used in [**91**] for a similar purpose; however, in that paper, the breach of scaling comes directly from the subcritical nature of the nonlinearity. An earlier related instance of this trick can be found in [**20**, §14].

Let u be a solution to (1.6) that obeys the hypotheses of Theorem 6.8. Let $\eta > 0$ be a small constant to be chosen later. Then by the almost periodicity modulo symmetries combined with (6.22), there exists $N_0 = N_0(\eta)$ such that

$$\|\nabla u_{\leq N_0}\|_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^d)} \leq \eta. \tag{6.23}$$

We turn now to our first step, that is, breaking scaling in a Lebesgue space. To this end, we define

$$A(N) := \begin{cases} N^{-\frac{2}{d-2}} \sup_{t\in\mathbb{R}} \|u_N(t)\|_{L_x^{\frac{2(d-2)}{d-4}}} & \text{for} \quad d \geq 6 \\ N^{-\frac12} \sup_{t\in\mathbb{R}} \|u_N(t)\|_{L_x^5} & \text{for} \quad d = 5. \end{cases}$$

for frequencies $N \leq 10N_0$. Note that by Bernstein's inequality combined with Sobolev embedding and (6.21),

$$A(N) \lesssim \|u_N\|_{L_t^\infty L_x^{\frac{2d}{d-2}}} \lesssim \|\nabla u\|_{L_t^\infty L_x^2} < \infty.$$

We next prove a recurrence formula for $A(N)$.

Lemma 6.9 (Recurrence)**.** *For all $N \leq 10N_0$,*

$$A(N) \lesssim_u \left(\tfrac{N}{N_0}\right)^\alpha + \eta^{\frac{4}{d-2}} \sum_{\frac{N}{10}\leq N_1\leq N_0} \left(\tfrac{N}{N_1}\right)^\alpha A(N_1) + \eta^{\frac{4}{d-2}} \sum_{N_1 < \frac{N}{10}} \left(\tfrac{N_1}{N}\right)^\alpha A(N_1),$$

where $\alpha := \min\{\frac{2}{d-2}, \frac12\}$.

Proof. We first give the proof in dimensions $d \geq 6$. Once this is completed, we will explain the changes necessary to treat $d = 5$.

Fix $N \leq 10N_0$. By time-translation symmetry, it suffices to prove

$$\begin{aligned} N^{-\frac{2}{d-2}} \|u_N(0)\|_{L_x^{\frac{2(d-2)}{d-4}}} \lesssim_u \left(\tfrac{N}{N_0}\right)^{\frac{2}{d-2}} &+ \eta^{\frac{4}{d-2}} \sum_{\frac{N}{10}\leq N_1\leq N_0} \left(\tfrac{N}{N_1}\right)^{\frac{2}{d-2}} A(N_1) \\ &+ \eta^{\frac{4}{d-2}} \sum_{N_1 < \frac{N}{10}} \left(\tfrac{N_1}{N}\right)^{\frac{2}{d-2}} A(N_1). \end{aligned} \tag{6.24}$$

Using the Duhamel formula (5.43) into the future followed by the triangle inequality, Bernstein, and the dispersive inequality, we estimate

$$\begin{aligned} N^{-\frac{2}{d-2}} \|u_N(0)\|_{L_x^{\frac{2(d-2)}{d-4}}} &\leq N^{-\frac{2}{d-2}} \Big\| \int_0^{N^{-2}} e^{-it\Delta} P_N F(u(t))\, dt \Big\|_{L_x^{\frac{2(d-2)}{d-4}}} \\ &\quad + N^{-\frac{2}{d-2}} \int_{N^{-2}}^\infty \big\| e^{-it\Delta} P_N F(u(t)) \big\|_{L_x^{\frac{2(d-2)}{d-4}}}\, dt \\ &\lesssim N \Big\| \int_0^{N^{-2}} e^{-it\Delta} P_N F(u(t))\, dt \Big\|_{L_x^2} \end{aligned}$$

$$\begin{aligned}
&+ N^{-\frac{2}{d-2}} \|P_N F(u)\|_{L_t^\infty L_x^{\frac{2(d-2)}{d}}} \int_{N^{-2}}^\infty t^{-\frac{d}{d-2}}\, dt \\
&\lesssim N^{-1} \|P_N F(u)\|_{L_t^\infty L_x^2} + N^{\frac{2}{d-2}} \|P_N F(u)\|_{L_t^\infty L_x^{\frac{2(d-2)}{d}}} \\
&\lesssim N^{\frac{2}{d-2}} \|P_N F(u)\|_{L_t^\infty L_x^{\frac{2(d-2)}{d}}}. \qquad (6.25)
\end{aligned}$$

Using the Fundamental Theorem of Calculus, we decompose

$$\begin{aligned}
F(u) = O(|u_{>N_0}||u_{\le N_0}|^{\frac{4}{d-2}}) &+ O(|u_{>N_0}|^{\frac{d+2}{d-2}}) + F(u_{\frac{N}{10}\le\cdot\le N_0}) \\
&+ u_{<\frac{N}{10}} \int_0^1 F_z\big(u_{\frac{N}{10}\le\cdot\le N_0} + \theta u_{<\frac{N}{10}}\big)\, d\theta \qquad (6.26)\\
&+ \overline{u_{<\frac{N}{10}}} \int_0^1 F_{\bar z}\big(u_{\frac{N}{10}\le\cdot\le N_0} + \theta u_{<\frac{N}{10}}\big)\, d\theta.
\end{aligned}$$

The contribution to the right-hand side of (6.25) coming from terms that contain at least one copy of $u_{>N_0}$ can be estimated in the following manner: Using Hölder, Bernstein, and (6.21),

$$\begin{aligned}
N^{\frac{2}{d-2}} \big\|P_N O(|u_{>N_0}||u|^{\frac{4}{d-2}})\big\|_{L_t^\infty L_x^{\frac{2(d-2)}{d}}} &\lesssim N^{\frac{2}{d-2}} \|u_{>N_0}\|_{L_t^\infty L_x^{\frac{2d(d-2)}{d^2-4d+8}}} \|u\|_{L_t^\infty L_x^{\frac{2d}{d-2}}}^{\frac{4}{d-2}} \\
&\lesssim_u N^{\frac{2}{d-2}} N_0^{-\frac{2}{d-2}}. \qquad (6.27)
\end{aligned}$$

Thus, this contribution is acceptable.

Next we turn to the contribution to the right-hand side of (6.25) coming from the last two terms in (6.26); it suffices to consider the first of them since similar arguments can be used to deal with the second.

Lemma A.13 yields

$$\big\|P_{>\frac{N}{10}} F_z(u)\big\|_{L_t^\infty L_x^{\frac{d-2}{2}}} \lesssim N^{-\frac{4}{d-2}} \|\nabla u\|_{L_t^\infty L_x^2}^{\frac{4}{d-2}}.$$

Thus, by Hölder's inequality and (6.23),

$$\begin{aligned}
N^{\frac{2}{d-2}} &\Big\|P_N\Big(u_{<\frac{N}{10}} \int_0^1 F_z\big(u_{\frac{N}{10}\le\cdot\le N_0} + \theta u_{<\frac{N}{10}}\big)\, d\theta\Big)\Big\|_{L_t^\infty L_x^{\frac{2(d-2)}{d}}} \\
&\lesssim N^{\frac{2}{d-2}} \|u_{<\frac{N}{10}}\|_{L_t^\infty L_x^{\frac{2(d-2)}{d-4}}} \Big\|P_{>\frac{N}{10}}\Big(\int_0^1 F_z\big(u_{\frac{N}{10}\le\cdot\le N_0} + \theta u_{<\frac{N}{10}}\big)\, d\theta\Big)\Big\|_{L_t^\infty L_x^{\frac{d-2}{2}}} \\
&\lesssim N^{-\frac{2}{d-2}} \|u_{<\frac{N}{10}}\|_{L_t^\infty L_x^{\frac{2(d-2)}{d-4}}} \|\nabla u_{\le N_0}\|_{L_t^\infty L_x^2}^{\frac{4}{d-2}} \\
&\lesssim \eta^{\frac{4}{d-2}} \sum_{N_1<\frac{N}{10}} \big(\tfrac{N_1}{N}\big)^{\frac{2}{d-2}} A(N_1). \qquad (6.28)
\end{aligned}$$

Hence, the contribution coming from the last two terms in (6.26) is acceptable.

We are left to estimate the contribution of $F(u_{\frac{N}{10}\le\cdot\le N_0})$ to the right-hand side of (6.25). We need only show

$$\|F(u_{\frac{N}{10}\le\cdot\le N_0})\|_{L_t^\infty L_x^{\frac{2(d-2)}{d}}} \lesssim \eta^{\frac{4}{d-2}} \sum_{\frac{N}{10}\le N_1\le N_0} N_1^{-\frac{2}{d-2}} A(N_1). \qquad (6.29)$$

As $d \geq 6$, we have $\frac{4}{d-2} \leq 1$. Using the triangle inequality, Bernstein, (6.23), and Hölder, we estimate as follows:

$$\begin{aligned}
&\|F(u_{\frac{N}{10}\leq\cdot\leq N_0})\|_{L_t^\infty L_x^{\frac{2(d-2)}{d}}} \\
&\lesssim \sum_{\frac{N}{10}\leq N_1\leq N_0} \big\|u_{N_1}|u_{\frac{N}{10}\leq\cdot\leq N_0}|^{\frac{4}{d-2}}\big\|_{L_t^\infty L_x^{\frac{2(d-2)}{d}}} \\
&\lesssim \sum_{\frac{N}{10}\leq N_1, N_2\leq N_0} \big\|u_{N_1}|u_{N_2}|^{\frac{4}{d-2}}\big\|_{L_t^\infty L_x^{\frac{2(d-2)}{d}}} \\
&\lesssim \sum_{\frac{N}{10}\leq N_1\leq N_2\leq N_0} \|u_{N_1}\|_{L_t^\infty L_x^{\frac{2(d-2)}{d-4}}} \|u_{N_2}\|_{L_t^\infty L_x^2}^{\frac{4}{d-2}} \\
&\quad + \sum_{\frac{N}{10}\leq N_2\leq N_1\leq N_0} \|u_{N_1}\|_{L_t^\infty L_x^2}^{\frac{4}{d-2}} \|u_{N_1}\|_{L_t^\infty L_x^{\frac{2(d-2)}{d-4}}}^{\frac{d-6}{d-2}} \|u_{N_2}\|_{L_t^\infty L_x^{\frac{2(d-2)}{d-4}}}^{\frac{4}{d-2}} \\
&\lesssim \sum_{\frac{N}{10}\leq N_1\leq N_2\leq N_0} \|u_{N_1}\|_{L_t^\infty L_x^{\frac{2(d-2)}{d-4}}} \eta^{\frac{4}{d-2}} N_2^{-\frac{4}{d-2}} \\
&\quad + \sum_{\frac{N}{10}\leq N_2\leq N_1\leq N_0} \eta^{\frac{4}{d-2}} N_1^{-\frac{4}{d-2}} \|u_{N_1}\|_{L_t^\infty L_x^{\frac{2(d-2)}{d-4}}}^{\frac{d-6}{d-2}} \|u_{N_2}\|_{L_t^\infty L_x^{\frac{2(d-2)}{d-4}}}^{\frac{4}{d-2}} \\
&\lesssim \eta^{\frac{4}{d-2}} \sum_{\frac{N}{10}\leq N_1\leq N_0} N_1^{-\frac{2}{d-2}} A(N_1) \\
&\quad + \eta^{\frac{4}{d-2}} \sum_{\frac{N}{10}\leq N_2\leq N_1\leq N_0} \big(\tfrac{N_2}{N_1}\big)^{\frac{16}{(d-2)^2}} \big(N_1^{-\frac{2}{d-2}} A(N_1)\big)^{\frac{d-6}{d-2}} \big(N_2^{-\frac{2}{d-2}} A(N_2)\big)^{\frac{4}{d-2}} \\
&\lesssim \eta^{\frac{4}{d-2}} \sum_{\frac{N}{10}\leq N_1\leq N_0} N_1^{-\frac{2}{d-2}} A(N_1).
\end{aligned}$$

This proves (6.29) and so completes the proof of the lemma in dimensions $d \geq 6$.

Consider now $d = 5$. Arguing as for (6.25), we have

$$N^{-\frac{1}{2}}\|u_N(0)\|_{L_x^5} \lesssim N^{\frac{1}{2}}\|P_N F(u)\|_{L_t^\infty L_x^{\frac{5}{4}}},$$

which we estimate by decomposing the nonlinearity as in (6.26). The analogue of (6.27) in this case is

$$N^{\frac{1}{2}}\|P_N O(|u_{>N_0}||u|^{\frac{4}{d-2}})\|_{L_t^\infty L_x^{\frac{5}{4}}} \lesssim N^{\frac{1}{2}}\|u_{>N_0}\|_{L_t^\infty L_x^{\frac{5}{2}}} \|u\|_{L_t^\infty L_x^{\frac{10}{3}}}^{\frac{4}{3}} \lesssim_u N^{\frac{1}{2}} N_0^{-\frac{1}{2}}.$$

Using Bernstein and Lemma A.11 together with (6.23), we replace (6.28) by

$$\begin{aligned}
&N^{\frac{1}{2}}\Big\|P_N\Big(u_{<\frac{N}{10}}\int_0^1 F_z\big(u_{\frac{N}{10}\leq\cdot\leq N_0} + \theta u_{<\frac{N}{10}}\big)\,d\theta\Big)\Big\|_{L_t^\infty L_x^{\frac{5}{4}}} \\
&\quad\lesssim N^{\frac{1}{2}}\|u_{<\frac{N}{10}}\|_{L_t^\infty L_x^5}\Big\|P_{>\frac{N}{10}}\Big(\int_0^1 F_z\big(u_{\frac{N}{10}\leq\cdot\leq N_0} + \theta u_{<\frac{N}{10}}\big)\,d\theta\Big)\Big\|_{L_t^\infty L_x^{\frac{5}{3}}} \\
&\quad\lesssim N^{-\frac{1}{2}}\|u_{<\frac{N}{10}}\|_{L_t^\infty L_x^5}\|\nabla u_{\leq N_0}\|_{L_t^\infty L_x^2}\|u_{\leq N_0}\|_{L_t^\infty L_x^{\frac{10}{3}}}^{\frac{1}{3}} \\
&\quad\lesssim \eta^{\frac{4}{3}} \sum_{N_1<\frac{N}{10}} \big(\tfrac{N_1}{N}\big)^{\frac{1}{2}} A(N_1).
\end{aligned}$$

Finally, arguing as for (6.29), we estimate

$$
\begin{aligned}
&\|F(u_{\frac{N}{10}\leq\cdot\leq N_0})\|_{L_t^\infty L_x^{\frac{5}{4}}} \\
&\lesssim \sum_{\frac{N}{10}\leq N_1,N_2\leq N_0} \big\|u_{N_1}u_{N_2}|u_{\frac{N}{10}\leq\cdot\leq N_0}|^{\frac13}\big\|_{L_t^\infty L_x^{\frac54}} \\
&\lesssim \sum_{\frac{N}{10}\leq N_1\leq N_2,N_3\leq N_0} \|u_{N_1}\|_{L_t^\infty L_x^5}\|u_{N_2}\|_{L_t^\infty L_x^{\frac{20}{9}}}\|u_{N_3}\|^{\frac13}_{L_t^\infty L_x^{\frac{20}{9}}} \\
&\quad + \sum_{\frac{N}{10}\leq N_3\leq N_1\leq N_2\leq N_0} \|u_{N_1}\|^{\frac23}_{L_t^\infty L_x^5}\|u_{N_1}\|^{\frac13}_{L_t^\infty L_x^{\frac{20}{9}}}\|u_{N_2}\|_{L_t^\infty L_x^{\frac{20}{9}}}\|u_{N_3}\|^{\frac13}_{L_t^\infty L_x^5} \\
&\lesssim \sum_{\frac{N}{10}\leq N_1\leq N_2,N_3\leq N_0} \|u_{N_1}\|_{L_t^\infty L_x^5}\eta N_2^{-\frac34}\eta^{\frac13}N_3^{-\frac14} \\
&\quad + \sum_{\frac{N}{10}\leq N_3\leq N_1\leq N_2\leq N_0} \|u_{N_1}\|^{\frac23}_{L_t^\infty L_x^5}\eta^{\frac13}N_1^{-\frac14}\eta N_2^{-\frac34}\|u_{N_3}\|^{\frac13}_{L_t^\infty L_x^5} \\
&\lesssim \eta^{\frac43}\sum_{\frac{N}{10}\leq N_1\leq N_0} N_1^{-\frac12}A(N_1) \\
&\quad + \eta^{\frac43}\sum_{\frac{N}{10}\leq N_3\leq N_1\leq N_0} \big(\tfrac{N_3}{N_1}\big)^{\frac13}\big(N_1^{-\frac12}A(N_1)\big)^{\frac23}\big(N_3^{-\frac12}A(N_3)\big)^{\frac13} \\
&\lesssim \eta^{\frac43}\sum_{\frac{N}{10}\leq N_1\leq N_0} N_1^{-\frac12}A(N_1).
\end{aligned}
$$

Putting everything together completes the proof of the lemma in the case $d=5$. □

This lemma leads very quickly to our first goal:

Proposition 6.10 (L_x^p breach of scaling)**.** *Let u be as in Theorem 6.8. Then*

$$u \in L_t^\infty L_x^p \quad \textit{for} \quad \tfrac{2(d+1)}{d-1} \leq p < \tfrac{2d}{d-2}. \tag{6.30}$$

In particular, by Hölder's inequality,

$$\nabla F(u) \in L_t^\infty L_x^r \quad \textit{for} \quad \tfrac{2(d-2)(d+1)}{d^2+3d-6} \leq r < \tfrac{2d}{d+4}. \tag{6.31}$$

Remark. As will be seen in the proof, p and r can be allowed to be smaller; however, the statement given will suffice for our purposes.

Proof. We only present the details for $d \geq 6$. The treatment of $d=5$ is completely analogous.

Combining Lemma 6.9 with Lemma A.15, we deduce

$$\|u_N\|_{L_t^\infty L_x^{\frac{2(d-2)}{d-4}}} \lesssim_u N^{\frac{4}{d-2}-} \quad \text{for all} \quad N \leq 10N_0. \tag{6.32}$$

In applying Lemma A.15, we set $N = 10\cdot 2^{-k}N_0$, $x_k = A(10\cdot 2^{-k}N_0)$, and take η sufficiently small.

By interpolation followed by (6.32), Bernstein, and (6.21),

$$
\begin{aligned}
\|u_N\|_{L_t^\infty L_x^p} &\leq \|u_N\|^{(d-2)(\frac12-\frac1p)}_{L_t^\infty L_x^{\frac{2(d-2)}{d-4}}}\|u_N\|^{\frac{d-2}{p}-\frac{d-4}{2}}_{L_t^\infty L_x^2} \\
&\lesssim_u N^{\frac{2(p-2)}{p}-}N^{\frac{d-4}{2}-\frac{d-2}{p}}
\end{aligned}
$$

$$\lesssim_u N^{\frac{1}{d+1}-}$$

for all $N \leq 10N_0$. Thus, using Bernstein together with (6.21), we obtain

$$\|u\|_{L_t^\infty L_x^p} \leq \|u_{\leq N_0}\|_{L_t^\infty L_x^p} + \|u_{>N_0}\|_{L_t^\infty L_x^p} \lesssim_u \sum_{N\leq N_0} N^{\frac{1}{d+1}-} + \sum_{N>N_0} N^{\frac{d-2}{2}-\frac{d}{p}} \lesssim_u 1,$$

which completes the proof of the proposition. □

Remark. With a few modifications, the argument used in dimension five can be adapted to dimensions three and four. However, while we may extend Proposition 6.10 in this way, $u(t,x) = W(x)$ provides an explicit counterexample to Theorem 6.8 in these dimensions. At a technical level, the obstruction is that the strongest dispersive estimate available is $|t|^{-d/2}$, which is insufficient to perform both integrals in the double Duhamel trick below when $d \leq 4$.

The second step is to use the double Duhamel trick to upgrade (6.30) to 'honest' negative regularity (i.e., in Sobolev sense). This will be achieved by repeated application of the following

Proposition 6.11 (Some negative regularity)**.** *Let $d \geq 5$ and let u be as in Theorem 6.8. Assume further that $|\nabla|^s F(u) \in L_t^\infty L_x^r$ for some $\frac{2(d-2)(d+1)}{d^2+3d-6} \leq r < \frac{2d}{d+4}$ and some $0 \leq s \leq 1$. Then there exists $s_0 = s_0(r,d) > 0$ such that $u \in L_t^\infty \dot H_x^{s-s_0+}$.*

PROOF. The proposition will follow once we establish

$$\big\||\nabla|^s u_N\big\|_{L_t^\infty L_x^2} \lesssim_u N^{s_0} \quad \text{for all} \quad N > 0 \quad \text{and} \quad s_0 := \tfrac{d}{r} - \tfrac{d+4}{2} > 0. \tag{6.33}$$

Indeed, by Bernstein combined with this and (6.21),

$$\begin{aligned}
\big\||\nabla|^{s-s_0+} u\big\|_{L_t^\infty L_x^2} &\leq \big\||\nabla|^{s-s_0+} u_{\leq 1}\big\|_{L_t^\infty L_x^2} + \big\||\nabla|^{s-s_0+} u_{>1}\big\|_{L_t^\infty L_x^2} \\
&\lesssim_u \sum_{N\leq 1} N^{0+} + \sum_{N>1} N^{(s-s_0+)-1} \\
&\lesssim_u 1.
\end{aligned}$$

Thus, we are left to prove (6.33). By time-translation symmetry, it suffices to prove

$$\big\||\nabla|^s u_N(0)\big\|_{L_x^2} \lesssim_u N^{s_0} \quad \text{for all} \quad N > 0 \quad \text{and} \quad s_0 := \tfrac{d}{r} - \tfrac{d+4}{2} > 0. \tag{6.34}$$

Using the Duhamel formula (5.43) both in the future and in the past, we write

$$\begin{aligned}
&\big\||\nabla|^s u_N(0)\big\|_{L_x^2}^2 \\
&\quad = \lim_{T\to\infty} \lim_{T'\to-\infty} \Big\langle i\int_0^T e^{-it\Delta} P_N |\nabla|^s F(u(t))\,dt, -i\int_{T'}^0 e^{-i\tau\Delta} P_N |\nabla|^s F(u(\tau))\,d\tau \Big\rangle \\
&\quad \leq \int_0^\infty \int_{-\infty}^0 \Big|\big\langle P_N|\nabla|^s F(u(t)), e^{i(t-\tau)\Delta} P_N |\nabla|^s F(u(\tau))\big\rangle\Big|\,dt\,d\tau.
\end{aligned}$$

We estimate the term inside the integrals in two ways. On one hand, using Hölder and the dispersive estimate,

$$\begin{aligned}
&\Big|\big\langle P_N|\nabla|^s F(u(t)), e^{i(t-\tau)\Delta} P_N |\nabla|^s F(u(\tau))\big\rangle\Big| \\
&\qquad \lesssim \big\|P_N|\nabla|^s F(u(t))\big\|_{L_x^r} \big\|e^{i(t-\tau)\Delta} P_N |\nabla|^s F(u(\tau))\big\|_{L_x^{r'}} \\
&\qquad \lesssim |t-\tau|^{\frac{d}{2}-\frac{d}{r}} \big\||\nabla|^s F(u)\big\|_{L_t^\infty L_x^r}^2.
\end{aligned}$$

On the other hand, using Bernstein,

$$\begin{aligned}\Big|\big\langle P_N|\nabla|^s F(u(t)), e^{i(t-\tau)\Delta}P_N|\nabla|^s F(u(\tau))\big\rangle\Big| &\lesssim \big\|P_N|\nabla|^s F(u(t))\big\|_{L^2_x}\big\|e^{i(t-\tau)\Delta}P_N|\nabla|^s F(u(\tau))\big\|_{L^2_x}\\ &\lesssim N^{2(\frac{d}{r}-\frac{d}{2})}\big\||\nabla|^s F(u)\big\|^2_{L^\infty_t L^r_x}.\end{aligned}$$

Thus,

$$\begin{aligned}\big\||\nabla|^s u_N(0)\big\|^2_{L^2_x} &\lesssim \big\||\nabla|^s F(u)\big\|^2_{L^\infty_t L^r_x}\int_0^\infty\int_{-\infty}^0 \min\{|t-\tau|^{-1}, N^2\}^{\frac{d}{r}-\frac{d}{2}}\,dt\,d\tau\\ &\lesssim N^{2s_0}\big\||\nabla|^s F(u)\big\|^2_{L^\infty_t L^r_x}.\end{aligned}$$

To obtain the last inequality we used the fact that $\frac{d}{r}-\frac{d}{2}>2$ since $r<\frac{2d}{d+4}$. Thus (6.34) holds, which finishes the proof of the proposition. □

Proof of Theorem 6.8. Proposition 6.10 allows us to apply Proposition 6.11 with $s=1$. We conclude that $u\in L^\infty_t \dot H^{1-s_0+}_x$ for some $s_0=s_0(r,d)>0$. Combining this with the fractional chain rule Lemma A.11 and (6.30), we deduce that $|\nabla|^{1-s_0+}F(u)\in L^\infty_t L^r_x$ for some $\frac{2(d-2)(d+1)}{d^2+3d-6}\le r<\frac{2d}{d+4}$. We are thus in the position to apply Proposition 6.11 again and obtain $u\in L^\infty_t\dot H^{1-2s_0+}_x$. Iterating this procedure finitely many times, we derive $u\in L^\infty_t\dot H^{-\varepsilon}_x$ for any $0<\varepsilon<s_0$.

This completes the proof of Theorem 6.8. □

6.4. Compactness in other topologies. In this subsection we show that solutions to the mass-critical NLS (or energy-critical NLS), which are solitons in the sense of Theorem 5.24 (or Theorem 5.25) and which enjoy sufficient additional regularity (or decay), have orbits that are not only precompact in L^2_x (or $\dot H^1_x$) but also in $\dot H^1_x$ (or L^2_x). Combining the two gives precompactness in H^1_x.

Lemma 6.12 (H^1_x compactness for the mass-critical NLS)**.** *Let $d\ge 1$ and let u be a soliton in the sense of Theorem 5.24. Assume further that $u\in L^\infty_t H^{1+\varepsilon}_x$ for some $\varepsilon=\varepsilon(d)>0$. Then for every $\eta>0$ there exists $C(\eta)>0$ such that*

$$\sup_{t\in\mathbb{R}}\int_{|x-x(t)|\ge C(\eta)}|\nabla u(t,x)|^2\,dx\lesssim_u \eta.$$

Remark. The hypotheses of Lemma 6.12 are known to be satisfied in dimensions $d\ge 2$ for spherically symmetric initial data; see [**43, 46**].

Proof. The entire argument takes place at a fixed t; in particular, we may assume $x(t)=0$.

First we control the contribution from the high frequencies. As $u\in L^\infty_t H^{1+\varepsilon}_x$ for some $\varepsilon>0$, then for any $R>0$,

$$\big\|\nabla u_{>N}(t)\big\|_{L^2_x(|x|\ge R)}\le\big\|\nabla u_{>N}(t)\big\|_{L^2_x}\lesssim N^{-\varepsilon}\big\||\nabla|^{1+\varepsilon}u\big\|_{L^\infty_t L^2_x}\lesssim_u N^{-\varepsilon}.$$

This can be made smaller than η by choosing $N=N(\eta)$ sufficiently large.

We now turn to the contribution coming from the low frequencies. A simple application of Schur's test reveals the following: For any $m\ge 0$,

$$\big\|\chi_{|x|\ge 2R}\nabla P_{\le N}\chi_{|x|\le R}\big\|_{L^2_x\to L^2_x}\lesssim_m N\langle RN\rangle^{-m}$$

uniformly in $R, N > 0$. Thus, by Bernstein's inequality,

$$\begin{aligned}\big\|\nabla u_{\leq N}(t)\big\|_{L^2_x(|x|\geq R)} &\\ &\leq \big\|\chi_{|x|\geq R}\nabla P_{\leq N}\chi_{|x|\leq R/2}u(t)\big\|_{L^2_x}+\big\|\chi_{|x|\geq R}\nabla P_{\leq N}\chi_{|x|\geq R/2}u(t)\big\|_{L^2_x}\\ &\lesssim_u N\langle RN\rangle^{-100} + N\|u(t)\|_{L^2_x(|x|\geq R/2)}.\end{aligned}$$

Choosing R sufficiently large (depending on N and η), we can ensure that the contribution of the low frequencies is less than η.

Combining the estimates for high and low frequencies yields the claim. □

We now turn our attention to the energy-critical NLS.

Lemma 6.13 (H^1_x compactness for the energy-critical NLS)**.** *Let $d \geq 3$ and let u be a soliton in the sense of Theorem 5.25 that belongs to $L^\infty_t \dot H^{-\varepsilon}_x$ for some $\varepsilon = \varepsilon(d) > 0$. Then for every $\eta > 0$ there exists $C(\eta) > 0$ such that*

$$\sup_{t\in\mathbb{R}} \int_{|x-x(t)|\geq C(\eta)} |u(t,x)|^2\,dx \lesssim_u \eta.$$

Remark. By Theorem 6.8, the hypotheses of this lemma are satisfied in dimensions $d \geq 5$.

Proof. The entire argument takes place at a fixed t; in particular, we may assume $x(t) = 0$.

First we control the contribution from the low frequencies: by hypothesis,

$$\big\|u_{<N}(t)\big\|_{L^2_x(|x|\geq R)}\leq \big\|u_{<N}(t)\big\|_{L^2_x}\lesssim N^\varepsilon\big\||\nabla|^{-\varepsilon}u\big\|_{L^\infty_t L^2_x}\lesssim_u N^\varepsilon.$$

This can be made smaller than η by choosing $N = N(\eta)$ small enough.

We now turn to the contribution from the high frequencies. A simple application of Schur's test reveals the following: For any $m \geq 0$,

$$\big\|\chi_{|x|\geq 2R}\Delta^{-1}\nabla P_{\geq N}\chi_{|x|\leq R}\big\|_{L^2_x\to L^2_x} \lesssim_m N^{-1}\langle RN\rangle^{-m}$$

uniformly in $R, N > 0$. On the other hand, by Bernstein,

$$\big\|\chi_{|x|\geq 2R}\Delta^{-1}\nabla P_{\geq N}\chi_{|x|\geq R}\big\|_{L^2_x\to L^2_x} \lesssim N^{-1}.$$

Together, these lead quickly to

$$\int_{|x|\geq 2R} |u_{\geq N}(t,x)|^2\,dx \lesssim N^{-2}\langle RN\rangle^{-100}\|\nabla u(t)\|^2_{L^2_x} + N^{-2}\int_{|x|\geq R}|\nabla u(t,x)|^2\,dx.$$

By choosing R large enough, we can render the first term smaller than η; the same is true of the second summand by virtue of $\dot H^1_x$-compactness:

$$\sup_{t\in\mathbb{R}}\int_{|x-x(t)|\geq C(\eta_1)} |\nabla u(t,x)|^2\,dx \leq \eta_1.$$

The lemma follows by combining our estimates for $u_{<N}$ and $u_{\geq N}$. □

7. Monotonicity formulae

The goal of this section it to introduce certain monotonicity formulae for the (non)linear Schrödinger equation. These have proved to be very powerful tools in the analysis of NLS; indeed, they have become *sine qua non* both for proving well-posedness and for describing the behaviour of solutions that blow up. Our goal here is just to give a small taste of what is available and how it can be used. Specific application to the mass- and energy- critical problems is discussed in Section 8.

7.1. The classical Virial theorem. Consider a classical mechanical system with n position coordinates, $q_1, \dots, q_n$, and n corresponding momenta, $p_1, \dots, p_n$. The energy is a sum of kinetic and potential terms,

$$H = K + V \quad \text{with} \quad K = \sum \tfrac{1}{2m_j} p_j^2 \quad \text{and} \quad V = V(q_1, \dots, q_n),$$

where m_j denote the mass of the particle associated to the jth coordinate. The basic precursor of all virial-like identities are the following simple calculations:

$$\tfrac{d}{dt} \sum \tfrac{1}{2} m_j q_j^2 = \sum m_j \dot{q}_j q_j = \sum p_j q_j, \tag{7.1}$$

$$\tfrac{d}{dt} \sum p_j q_j = \sum p_j \dot{q}_j + \dot{p}_j q_j = \sum \tfrac{1}{m_j} p_j^2 - \tfrac{\partial V}{\partial q_j} q_j. \tag{7.2}$$

Theorem 7.1 (The Virial Theorem of Clausius, [**17**])**.** *If V is a homogeneous function of degree k, then the time averages of kinetic and potential energies are related by $\langle K \rangle = \frac{k}{2} \langle V \rangle$ along any orbit that remains inside a compact set in phase space. More precisely,*

$$\frac{1}{2T} \int_{-T}^{T} \Big[\sum \tfrac{1}{2m_j} p_j^2(t) - \tfrac{k}{2} V(q_1(t), \dots, q_n(t)) \Big] \, dt = O(\tfrac{1}{T}) \tag{7.3}$$

as $T \to \infty$.

Proof. The result follows quickly from (7.2) together with

$$\sum \tfrac{\partial V}{\partial q_j} q_j = kV,$$

which is a consequence of the homogeneity of V. □

Remark. The quantity $\sum \dot{p}_j q_j$ (or rather, its time average) is known as the *virial*. The name was coined by Clausius and derives from the Latin for 'force'. A more famous notion (and name) due to Clausius is 'entropy'. His nomenclature for kinetic energy, 'vis viva', and potential energy, 'ergal', however, did not catch on.

Example 7.1. For gravitational attraction, the potential energy is homogeneous of degree -1. Thus, for the eight major planets (whose orbits are approximately circular), the virial theorem gives a relation between the orbital radius r and the orbital velocity v of the form $v^2 = GM/r$, where M is the solar mass and G is the gravitational constant. As the orbital period is given by $T = 2\pi r/v$, we obtain Kepler's third law: T^2/r^3 is the same for all the major planets. Indeed, we find that this constant is $4\pi^2/GM = 3.0 \times 10^{-19} s^2 m^{-3}$, which agrees with astronomical data.

Example 7.2 (Weighing things in space)**.** Through a telescope, one may approximately measure lengths and speeds (Doppler effect). Now consider applying the virial identity to some form of self-gravitating ensemble of similar objects (e.g., stars or galaxies). The potential energy is quadratic in the mass, while the kinetic energy is linear in the mass. Given the typical distances involved and the typical speeds involved, one can quickly pop out a crude estimate for the total mass.

7.2. Some Lyapunov functions. In the field of ordinary differential equations, functions that are monotone in time (under the flow) are traditionally referred to as Lyapunov functions, in honour of the important work of A. M. Lyapunov on stability. Our applications of monotonicity formulae are perhaps better described as instability. The following two examples convey something of the spirit of this.

Example 7.3. Consider a particle in $\mathbb{R}^3\setminus\{0\}$ moving in the presence of a repulsive potential $V(q)$, for example, $V(q)=|q|^{-1}$. The word repulsive is meant in the technical sense that $q\cdot\nabla V(q)<0$, which says that the radial component of the force on the particle always points away from the origin. By referring to (7.2), we see that $\sum p_j q_j$ is strictly increasing (in time) along any trajectory of the system. We immediately see that there can be no periodic orbits; indeed, any orbit must escape to (spatial) infinity as $t\to\pm\infty$.

Example 7.4. If we choose $m_j\equiv 1$ and $V(q)=-|q|^{-2}$, then (7.1) and (7.2) become

$$\tfrac{d^2}{dt^2}\,\tfrac12|q|^2=2H(p,q).$$

If the initial energy is negative, then $|q(t)|^2$ is a concave function of time. It is also non-negative. Thus we see that the particle falls into the origin in finite time.

In this section, we will discuss Lyapunov functionals for the flow

$$iu_t=-\Delta u+Vu+\mu|u|^p u. \tag{7.4}$$

We need only consider as potential Lyapunov functionals those which are odd under time reversal; even functionals, at least, cannot be monotone. Probably the simplest example is the quadratic form associated to a self-adjoint differential operator of first order:

$$\begin{aligned} F(u) &:= \tfrac1i\int_{\mathbb{R}^d}\bar u(x)\big[a_j(x)\partial_j+\partial_j a_j(x)\big]u(x)\,dx\\ &=2\int_{\mathbb{R}^d}a_j(x)\operatorname{Im}\big(\bar u(x)\partial_j u(x)\big)\,dx, \end{aligned} \tag{7.5}$$

where a_j are real-valued functions on $\mathbb{R}^d$ and (both here and below) the repeated index j is summed over $1\le j\le d$. As we will only consider cases where $F(u)$ has spherical symmetry, we are guaranteed that there is a function $a(x)$ so that $a_j(x)=\partial_j a(x)$. This restriction has the happy consequence that we may use subscripts to denote partial derivatives, which we shall do from now on. A more scientific consequence is the first part of the following:

Lemma 7.2 (Morawetz/Virial identity)**.** *Under the flow* (7.4)*,*

$$F(u)=\frac{d}{dt}\int_{\mathbb{R}^d}a(x)|u(t,x)|^2\,dx \tag{7.6}$$

$$\frac{d}{dt}F(u(t))=\int_{\mathbb{R}^d}-a_{jjkk}|u|^2+4a_{jk}\bar u_j u_k+\mu\tfrac{2p}{p+2}a_{jj}|u|^{p+2}-2a_jV_j|u|^2. \tag{7.7}$$

Here (as always in this subsection) subscripts indicate partial derivatives and repeated indices are summed.

We will discuss three applications in approximately historical order. Our first relates to the spectral and scattering theory of the linear Schrödinger equation and can be viewed as a quantum version of Example 7.3. Earlier still, identities analogous to (7.7) played an important role in the problem of obstacle scattering for the linear wave equation. Identities of this type are commonly known as Morawetz identities in honour of her pioneering work in this direction; see [**53**] for the link to scattering theory and [**60**] for an early retrospective.

Before discussing the linear Schrödinger equation, we first wish to present some completely abstract results about Lyapunov functions in quantum mechanics. The

Putnam of the first theorem is *not* that of the competition; the name of the second theorem was coined in [**70**] and reflects the initials of Ruelle, Amrein, Georgescu, and Enss, rather than any ill-feeling.

Theorem 7.3 (Putnam–Kato Theorem, [**36, 69**])**.** *Let H and A be bounded self-adjoint operators on a Hilbert space. If $C := i[H, A]$ is positive definite, then H has purely absolutely continuous spectrum.*

Remark. Under certain technical assumptions, one may allow H and/or A to be unbounded; indeed, in the PDE context, this is the most common situation. However, our goal here is simply to give a taste of what may be expected.

Proof. As A is bounded, we can quickly see that $\langle e^{-itH}\phi, Ce^{-itH}\phi\rangle$ belongs to $L^1_t(\mathbb{R})$ for all vectors ϕ. Thus, for all vectors ϕ in the range of $\sqrt{C}$, which is dense in the Hilbert space, we have $\langle \phi, e^{-itH}\phi\rangle \in L^2_t(\mathbb{R})$. The result now follows from the fact that only absolutely continuous measures can have square integrable Fourier transforms (cf. Parseval's Theorem). □

Theorem 7.4 (RAGE Theorem)**.** *Let H be a self-adjoint operator with purely absolutely continuous spectrum and let C be a bounded self-adjoint operator with $C(H-i)^{-1}$ compact. Then*

$$\langle e^{-itH}\phi,\, Ce^{-itH}\phi\rangle \to 0, \quad \textit{as } t \to \pm\infty,$$

for all ϕ in the Hilbert space. If H has purely continuous spectrum, then

$$\frac{1}{2T}\int_{-T}^{T} \langle e^{-itH}\phi,\, Ce^{-itH}\phi\rangle\, dt \to 0, \quad \textit{as } T \to \pm\infty.$$

Proof. The results follow (respectively) from the Riemann–Lebesgue lemma and Wiener's lemma,

$$\frac{1}{2T}\int_{-T}^{T} \left|\int e^{-i\omega t}\, d\mu(\omega)\right|^2 dt \longrightarrow \sum_{\omega\in\mathbb{R}} \big|\mu(\{\omega\})\big|^2 \quad \text{as } T \to \infty,$$

after first applying the spectral theorem. □

The connection of Theorem 7.3 to Lyapunov functions is clear. We have included Theorem 7.4 to convey the fact that Theorem 7.3 guarantees that all trajectories escape to infinity in a fairly strong sense; indeed one may deduce the following from the RAGE Theorem:

Exercise. Suppose H is a self-adjoint operator and ϕ a vector in the associated Hilbert space. Show that the orbit $\{e^{-itH}\phi : t \in \mathbb{R}\}$ is pre-compact if and only if ϕ is a linear combination of eigenvectors of H, that is, if and only if the spectral measure associated to (H, ϕ) is of pure-point type.

Finally, we turn to our long-promised application to the linear Schrödinger equation. What we present is a special case of results contained in two early papers of R. Lavine, [**51, 52**]. This material is also discussed at some length in [**71**, §XIII.7]. Note that our particular statement has been chosen to simplify the exposition and in no way represents the limit of the method.

Theorem 7.5. *Suppose $d \geq 3$ and $V : \mathbb{R}^d \to \mathbb{R}$ obeys $|V(x)| \lesssim \langle x\rangle^{-1-\varepsilon}$ and is repulsive in the sense that $x \cdot \nabla V \leq 0$ as a distribution. Then $H := -\Delta + V$ has purely absolutely continuous spectrum. Moreover, the limits $\lim_{t\to\pm\infty} e^{-it\Delta}e^{-itH}$ and $\lim_{t\to\pm\infty} e^{itH}e^{it\Delta}$ exist in the strong topology and define unitary operators.*

Proof. We will prove absolute continuity by adapting the argument used to prove Theorem 7.3. For the scattering results, see the references given above.

Set $a(x) = \langle x \rangle$. For $\phi \in C_c^\infty(\mathbb{R}^d)$, let $u(t) := e^{-itH}\phi$. Then by (7.7),

$$\frac{d}{dt}F(u(t)) \geq \int_{\mathbb{R}^d} |u(t,x)|^2[-\Delta\Delta a](x)\,dx \gtrsim \int_{\mathbb{R}^d} |u(t,x)|^2 \langle x \rangle^{-7}\,dx. \tag{7.8}$$

Note that the missing terms have the right sign for the following reasons: a is convex, so a_{jk} is a positive definite matrix; μ is zero since we consider the linear equation; the potential is assumed repulsive.

Now, mass/energy conservation guarantee that $u \in L_t^\infty H_x^1$, which then implies that $F(u)$ is bounded. Integrating (7.8) in time and using $\phi \in L^2(\langle x \rangle^7\,dx)$, we may deduce that $\langle \phi, e^{-itH}\phi \rangle \in L^2(dt)$. This proves that the spectral measure associated to (H,ϕ) is absolutely continuous (via Parseval's theorem) for a dense set of $\phi \in L_x^2(\mathbb{R}^d)$. Thus, we may conclude that H has purely absolutely continuous spectrum. □

Before turning to the nonlinear Schrödinger equation, we wish to draw the readers attention to two further developments connected to the material just described. The first is Mourre's method, which extends and refines the ideas behind the proof of Theorem 7.5. This is surveyed in [**22**, Ch. 4]. Chapter 5 of that book describes the Enss method in scattering theory. The idea here is that because of the RAGE Theorem, any part of the solution not described by bound states must travel far from the (spatial) origin. Once far away, the wave packet will continue to move outward since the potential is very weak out there. Parts of the argument in [**43**] can be viewed as an NLS incarnation of the Enss approach.

Our first NLS application of the Morawetz/Virial identity is an analogue of Example 7.4 and shows that for certain initial data, the solution of NLS must blow up in finite time. This is the well-known concavity argument; see, for instance, [**31, 102**]:

Theorem 7.6 (Finite-time blow up)**.** *Consider*

$$iu_t = -\Delta u - |u|^p u \quad \textit{with} \quad \tfrac{4}{d} \leq p \leq \tfrac{4}{d-2}. \tag{7.9}$$

Initial data $u_0 \in \Sigma := \{f \in H_x^1(\mathbb{R}^d) : |x|f \in L_x^2(\mathbb{R}^d)\}$ *with negative energy (that is,* $E(u_0) < 0$*) lead to solutions which blow up in finite time in both the past and future.*

Remark. Such negative energy initial data do exist. Indeed, if $f \in \Sigma$ is nonzero then $u_0 = \lambda f$ will have negative energy for λ sufficiently large, because the kinetic and potential energies contain different powers of u_0. By the same reasoning, $E(u_0) > 0$ for small initial data.

Proof. By the local theory discussed in Section 3, the H_x^1 norm will remain finite (though not necessarily uniformly bounded!) for as long as the solution exists. Choosing $a(x) = |x|^2$ in (7.6) gives

$$\frac{d}{dt}\int_{\mathbb{R}^d} |x|^2|u(t,x)|^2\,dx = 4\int_{\mathbb{R}^d} \operatorname{Im}\big(\bar{u}(x)\; x \cdot \nabla u(x)\big)\,dx = O(\|\nabla u\|_{L_x^2}\|xu\|_{L_x^2}), \tag{7.10}$$

which shows that the the second moment will also remain finite throughout the lifespan of the solution. More importantly, (7.7) from Lemma 7.2 shows that

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^d} |x|^2|u(t,x)|^2\,dx = \int_{\mathbb{R}^d} 8|\nabla u(t,x)|^2 - \tfrac{4pd}{p+2}|u(t,x)|^{p+2}\,dx \tag{7.11}$$

$$= 16E(u_0) - \int_{\mathbb{R}^d} \tfrac{4(pd-4)}{p+2} |u(t,x)|^{p+2}\, dx. \tag{7.12}$$

Thus (using the conservation and negativity of energy) we see that a manifestly positive quantity is trapped beneath an inverted parabola, at least on the lifespan of the solution. This guarantees that the lifespan must be finite in both time directions. □

There are two natural directions to try to extend Theorem 7.6. The first is to weaken the hypothesis $u_0 \in \Sigma$; indeed, it certainly seems reasonable to imagine that the result still holds for negative energy data $u_0 \in H^1_x$. At present this is only known under the additional assumption that u_0 is spherically symmetric; see [**65**] where this is proved for $4/d \le p < \min\{4, 4/(d-2)\}$ and $d \ge 2$. Secondly, one might hope to take advantage of the second term on the right-hand side of (7.12) to prove finite-time blowup for certain positive energy initial data. This is indeed possible:

Exercise ([**38**, Remark 3.14])**.** Use Theorem 4.4 to prove the following in the energy-critical case: if $E(u_0) < E(W)$ then RHS(7.11) cannot change sign. In particular, if $u_0 \in \Sigma$, $E(u_0) < E(W)$, and RHS(7.11) is negative for u_0, then the solution will blow up in finite time.

Combining this with the argument in [**65**], one may show that if $u_0 \in H^1_{\mathrm{rad}}$, $E(u_0) < E(W)$, and RHS(7.11) is negative for u_0, then the solution will blow up in finite time; for complete details see [**44**]. Analogous arguments in the subcritical case can be found in [**34**].

The first application of Lemma 7.2 to the scattering problem for NLS appears to be [**55**], although the authors freely acknowledge their debt to earlier work on the nonlinear Klein–Gordon equation, [**60, 61**]. This innovation led to considerable developments in the scattering theory for the energy-subcritical (but mass-supercritical) defocusing problem, particularly at the hands of Ginibre and Velo; see [**29**], for example, and the references therein.

The Morawetz identity also played a very important role in the first treatment of the large-data energy-critical problem [**7**]; this was for spherically symmetric data:

Proposition 7.7 (Morawetz à la Bourgain, [**7**])**.** *Let u be a spherically symmetric solution to the defocusing energy-critical NLS on a spacetime slab $I \times \mathbb{R}^d$. Then, for any $K \ge 1$, we have*

$$\int_I \int_{|x| \le K|I|^{\frac{1}{2}}} \frac{|u(t,x)|^{\frac{2d}{d-2}}}{|x|}\, dx\, dt \lesssim K|I|^{\frac{1}{2}} E(u). \tag{7.13}$$

In particular, for this NLS there are no solitons or low-to-high cascades, in the sense of Theorem 5.25.

Proof. The inequality (7.13) follows (with a little work) from Lemma 7.2 with $a(x) := R\,\psi\big(\frac{x}{R}\big)$, provided we take $R = K|I|^{1/2}$ and choose $\psi(x)$ to be a spherically symmetric nondecreasing (in radius) function obeying

$$\psi(x) = \begin{cases} |x| & \text{if } |x| \le 1 \\ \frac{3}{2} & \text{if } |x| \ge 2, \end{cases}$$

which is smooth except at the origin.

We now turn our attention to the second assertion. By Lemma 5.18, we may partition $\mathbb{R}$ into intervals I_j so that for some $t_j \in I_j$ we have $|I_j| \sim_u N(t_j)^{-2}$ and $N(t) \sim_u N(t_j)$ for all $t \in I_j$. Let I be the union of some contiguous sub-collection of the intervals I_j. Then, using almost periodicity, (7.13) implies

$$\int_I N(t)\,dt \lesssim_u |I|^{\frac{1}{2}} E(u). \tag{7.14}$$

This shows that $N(t)$ must go to zero rather quickly; it is certainly inconsistent with the scenarios mentioned in the proposition. □

Bourgain's argument [**7**] was simplified and extended in [**89**], which also obtains a much better spacetime bound. See also [**45**], which incorporates some further simplifications made possible by Lemma A.12.

The papers just referenced do not discuss almost periodic solutions, nor did the extraction of the three enemies (Theorem 5.25) exist at that time. It was however known that solutions with large Strichartz norm must regularly contain bubbles of energy concentration; the natural analogue of $N(t)$ is the reciprocal of the characteristic length scale of these bubbles. Following [**89**], the Morawetz inequality was used roughly as follows: by making the most of (7.14), it is shown that there must be a cascade of bubbles of rapidly changing size in a comparatively small amount of time. This is then contradicted using the almost conservation of mass in finite regions.

With the exception of Theorem 7.6, the applications of Lemma 7.2 that we have discussed so far have discarded the kinetic term $a_{jk}u_j\bar{u}_k$. Indeed, as long as a is a convex function, it will have a favourable sign. By choosing a slightly more convex a, one may exhibit a weighted version of the kinetic energy. This non-linear analogue of local smoothing (cf. Proposition 4.14) has proved valuable in the treatment of the mass-critical NLS, at least, for spherically symmetric data; see [**97**].

Exercise (See [**90**, p. 87])**.** Let u be a solution of (7.4) in three or more dimensions with $V \equiv 0$ and $\mu \geq 0$. By using Lemma 7.2 with $a(x) = \langle x\rangle - \varepsilon\langle x\rangle^{1-\varepsilon}$, show that

$$\int_I \int_{\mathbb{R}^d} |\nabla u(t,x)|^2 \langle x\rangle^{-1-\varepsilon}\,dx\,dt \lesssim \|u\|_{L_t^\infty L_x^2} \|\nabla u\|_{L_t^\infty L_x^2}.$$

In fact, (a further exercise) the right-hand side can be upgraded to $\||\nabla|^{1/2}u\|_{\infty,2}^2$.

The restriction to dimensions three and higher stems from the lack of a good choice for a in one and two dimensions, that is, of a convex a with a_k bounded and $-\Delta\Delta a$ positive.

7.3. Interaction Morawetz. The weight appearing in (7.13) is strongly tied to the case of spherically symmetric data. In [**19**], a variant of the Morawetz identity was introduced that is better adapted to the treatment of general (not spherically symmetric) data. This is the topic of this subsection.

One of the early applications of the new monotonicity formula was to the proof of global well-posedness and scattering for the three dimensional energy-critical defocusing nonlinear Schrödinger equation, [**20**]. This argument was subsequently adapted to four dimensions, [**75**], and then to dimensions five and higher, [**103, 104**].

In the papers just mentioned, it was necessary to introduce a frequency cutoff; this means that one needs to consider solutions to an inhomogeneous NLS:

$$iu_t = -\Delta u + |u|^p u + F, \tag{7.15}$$

where F is some function of space and time. Note that we limit ourselves to the defocusing case, since this is where the interaction Morawetz identity has proved most useful.

Beginning with (7.15), a few elementary computations reveal

$$\partial_t |u|^2 = -2\,\mathrm{Im}(u_k \bar{u})_k + 2\,\mathrm{Im}(F\bar{u}) \tag{7.16}$$

$$\partial_t 2\,\mathrm{Im}(u_k\bar{u}) = \Delta\big(|u|^2\big)_k - 4\,\mathrm{Re}\big(\bar{u}_k u_j\big)_j - \tfrac{2p}{p+2}\big(|u|^{p+2}\big)_k + 2\,\mathrm{Re}\big(u_k\bar{F} - F_k\bar{u}\big). \tag{7.17}$$

As in the previous subsection, subscripts denote spatial derivatives and repeated indices are summed.

Proposition 7.8 (Interaction Morawetz, [**19**])**.** *If u obeys* (7.15) *and*

$$M(t) := 2\iint_{\mathbb{R}^d\times\mathbb{R}^d} |u(y)|^2 a_k(x-y)\,\mathrm{Im}\{u_k(x)\bar{u}(x)\}\,dx\,dy, \tag{7.18}$$

for some even convex function $a:\mathbb{R}^d\to\mathbb{R}$, then

$$\begin{aligned}\partial_t M(t) \geq \iint_{\mathbb{R}^d\times\mathbb{R}^d}\Big\{ &- a_{kkjj}(x-y)\,|u(y)|^2|u(x)|^2 + \tfrac{2p}{p+2} a_{kk}(x-y)|u(x)|^{p+2}|u(y)|^2 \\ &+ 2a_k(x-y)|u(y)|^2\,\mathrm{Re}\big[u_k(x)\bar{F}(x) - F_k(x)\bar{u}(x)\big] \\ &+ 4a_k(x-y)(\mathrm{Im}\,F(y)\bar{u}(y))(\mathrm{Im}\,u_k(x)\bar{u}(x))\Big\}\,dx\,dy. \end{aligned} \tag{7.19}$$

Proof. Patient computation shows that with the addition of one term, (7.19) would become an equality. In this way, one sees that the claim is equivalent to

$$4\iint_{\mathbb{R}^d\times\mathbb{R}^d} a_{jk}(x-y)\big[|u(y)|^2\bar{u}_j(x)u_k(x) - (\mathrm{Im}\,\bar{u}(y)u_j(y))(\mathrm{Im}\,\bar{u}(x)u_k(x))\big]\,dx\,dy \geq 0,$$

which is what we will explain here.

Fix x and y. As a is convex, the matrix $a_{jk}(x-y)$ is positive semi-definite. Now suppose e is one of the eigenvectors of this matrix. By elementary considerations,

$$\begin{aligned}\big|e_k e_j(\mathrm{Im}\,\bar{u}(y)u_j(y))(\mathrm{Im}\,\bar{u}(x)u_k(x))\big| &\leq |u(y)|\,|e\cdot\nabla u(y)|\,|u(x)|\,|e\cdot\nabla u(x)| \\ &\leq \tfrac{1}{2}|u(x)|^2|e\cdot\nabla u(y)|^2 + \tfrac{1}{2}|u(y)|^2|e\cdot\nabla u(x)|^2.\end{aligned}$$

Writing out $a_{jk}(x-y)$ in terms of its eigenvalues and vectors, this shows that the integrand is indeed non-negative, at least, after symmetrization under $x\leftrightarrow y$. □

Exercise (See [**19**]). Show that for $d=3$ and $a(x)=|x|$, Lemma 7.8 implies

$$\int_{\mathbb{R}}\int_{\mathbb{R}^3} |u(t,x)|^4\,dx\,dt \lesssim \|u\|^3_{L^\infty_t L^2_x}\|\nabla u\|_{L^\infty_t L^2_x}$$

for solutions of (7.15) with $F\equiv 0$.

In dimensions $d\geq 4$, there is an analogous result although the left-hand side takes a much less simple form. Nevertheless, it allows one to deduce the following:

Proposition 7.9. *For $d\geq 3$ and $F\equiv 0$, any solution to* (7.15) *obeys*

$$\|u\|_{L^{d+1}_t L^{\frac{2(d+1)}{d-1}}_x(I\times\mathbb{R}^d)} \lesssim \|u\|_{L^\infty_t H^{1}_x(I\times\mathbb{R}^d)}.$$

As noted above, this is in [**19**] when $d = 3$. For $d \geq 4$, the result appears as [**95**, Proposition 5.1]; see also [**103**, §5], which uses the same ideas. One application of this lemma given in [**95**] is a simplified proof of scattering for defocusing inter-critical NLS. The original proof by Ginibre and Velo, [**29**], used the standard (Lin–Strauss) Morawetz identity.

As noted at the end of the previous section, there are some difficulties in using the standard Morawetz estimate in one and two dimensions. Some of these difficulties can be alleviated by switching to the interaction Morawetz estimate. See for instance [**67**]. There is also a four-particle interaction Morawetz that has proved effective in the one-dimensional setting:

Proposition 7.10 ([**18**, Proposition 3.1])**.** *Let u be a solution to a defocusing NLS in one space dimension, then*

$$\int_I \int_{\mathbb{R}} |u(t,x)|^8 \, dx \, dt \lesssim \|u\|^2_{L_t^\infty \dot H_x^{1/2}(I\times\mathbb{R})} \|u_0\|^6_{L_t^\infty L_x^2(I\times\mathbb{R})}. \tag{7.20}$$

For a recent review of interaction Morawetz inequalities and their application to the scattering problem for inter-critical NLS see [**30**].

8. Nihilism

In this section we use conservation laws and monotonicity formulae to preclude the global enemies described in Theorems 5.24 and 5.25, provided that these enemies obey additional regularity/decay. More precisely, we show how to dispense with soliton and frequency cascade solutions that belong to $L_t^\infty H_x^{1+\varepsilon}$ for some $\varepsilon > 0$ in the mass-critical case or to $L_t^\infty \dot H_x^{-\varepsilon}$ in the energy-critical case. Recall that in the mass-critical case, the spherically symmetric soliton and cascade were shown to enjoy such additional regularity in [**43, 46**] for $d \geq 2$. For the energy-critical NLS, Theorem 6.8 established the decay needed in dimensions $d \geq 5$.

We remind the reader that enemies which are not global, that is, the self-similar solution (in the mass-critical case) or the finite-time blowup solution (in the energy-critical case) can be precluded via more direct techniques. In the former case it is sufficient to prove $u(t) \in H_x^1$ for some $t \in (0,\infty)$, since then the global theory for H_x^1 initial data leads to a contradiction. Theorem 6.1 establishes this for spherically symmetric initial data and $d \geq 2$.

For the energy-critical NLS, finite-time blowup solutions (as described in Theorem 5.25) were precluded in Theorem 6.7 for all dimensions $d \geq 3$.

8.1. Frequency cascade solutions.

We first turn our attention to high-to-low frequency cascade solutions of the mass-critical NLS (cf. Theorem 5.24). We will show that no such solutions may belong to $L_t^\infty H_x^{1+\varepsilon}$ for some $\varepsilon > 0$. We would like to point out that regularity above H_x^1 is needed for the argument we present below.

Theorem 8.1 (Absence of mass-critical cascades)**.** *Let $d \geq 1$. There are no non-zero global solutions to* (1.4) *which are double high-to-low frequency cascades in the sense of Theorem 5.24 and which obey $u \in L_t^\infty H_x^{1+\varepsilon}$ for some $\varepsilon = \varepsilon(d) > 0$.*

Proof. Suppose to the contrary that there is such a solution u. Using a Galilean transformation, we may set its momentum equal to zero, that is,

$$\int_{\mathbb{R}^d} \xi |\hat u(t,\xi)|^2 \, d\xi = 0.$$

Note that u remains in $L_t^\infty H_x^{1+\varepsilon}$.

By hypothesis $u \in L_t^\infty H_x^1$ and so the energy

$$E(u) = E(u(t)) = \int_{\mathbb{R}^d} \tfrac{1}{2}|\nabla u(t,x)|^2 + \mu \tfrac{d}{2(d+2)} |u(t,x)|^{\frac{2(d+2)}{d}}\, dx$$

is finite and conserved. Moreover, as $M(u) < M(Q)$ in the focusing case, the sharp Gagliardo-Nirenberg inequality gives

$$\|\nabla u(t)\|_{L_x^2(\mathbb{R}^d)}^2 \sim_u E(u) \sim_u 1 \tag{8.1}$$

for all $t \in \mathbb{R}$. We will now reach a contradiction by proving that $\|\nabla u(t)\|_2 \to 0$ along any sequence where $N(t) \to 0$. The existence of two such time sequences is guaranteed by the fact that u is a double high-to-low frequency cascade.

Let $\eta > 0$ be arbitrary. By Definition 5.1, we can find $C(\eta) > 0$ such that

$$\int_{|\xi - \xi(t)| \geq C(\eta) N(t)} |\hat{u}(t,\xi)|^2\, d\xi \leq \eta^2$$

for all t. Meanwhile, by hypothesis, $u \in L_t^\infty H_x^{1+\varepsilon}(\mathbb{R} \times \mathbb{R}^d)$ for some $\varepsilon > 0$. Thus,

$$\int_{\mathbb{R}^d} |\xi|^{2+2\varepsilon} |\hat{u}(t,\xi)|^2\, d\xi \lesssim_u 1$$

for all t. Therefore, combining the two estimates gives

$$\int_{|\xi - \xi(t)| \geq C(\eta) N(t)} |\xi|^2 |\hat{u}(t,\xi)|^2\, d\xi \lesssim_u \eta^{\frac{2\varepsilon}{1+\varepsilon}}.$$

On the other hand, from mass conservation and Plancherel's theorem we have

$$\int_{|\xi - \xi(t)| \leq C(\eta) N(t)} |\xi|^2 |\hat{u}(t,\xi)|^2\, d\xi \lesssim_u \big[C(\eta) N(t) + |\xi(t)|\big]^2.$$

Summing these last two bounds and using Plancherel's theorem again, we obtain

$$\|\nabla u(t)\|_{L_x^2(\mathbb{R}^d)} \lesssim_u \eta^{\frac{\varepsilon}{1+\varepsilon}} + C(\eta) N(t) + |\xi(t)|$$

for all t. As u is a double high-to-low frequency cascade, there exists a sequence of times $t_n \to \infty$ such that $N(t_n) \to 0$. As $\eta > 0$ is arbitrary, it remains to prove that $|\xi(t_n)| \to 0$ as $n \to \infty$ in order to deduce $\|\nabla u(t_n)\|_2 \to 0$, which would contradict (8.1), thus concluding the proof of the theorem.

To see that $|\xi(t_n)| \to 0$ as $n \to \infty$ we use mass conservation, the uniform $H_x^{1/2+\varepsilon}$ bound for some $\varepsilon > 0$, and the fact that $N(t_n) \to 0$, together with the vanishing of the total momentum of u. □

We now turn our attention to the energy-critical NLS and preclude low-to-high frequency cascade solutions belonging to $L_t^\infty \dot{H}_x^{-\varepsilon}$ for some $\varepsilon > 0$.

Theorem 8.2 (Absence of energy-critical cascades)**.** *Let $d \geq 3$. There are no non-zero global solutions to* (1.6) *that are low-to-high frequency cascades in the sense of Theorem 5.25 and that belong to $L_t^\infty \dot{H}_x^{-\varepsilon}$ for some $\varepsilon > 0$.*

Proof. Suppose for a contradiction that there existed such a solution u. Then by hypothesis, $u \in L_t^\infty L_x^2$; thus, by the conservation of mass,

$$0 < M(u) = M(u(t)) = \int_{\mathbb{R}^d} |u(t,x)|^2\, dx < \infty \quad \text{for all} \quad t \in \mathbb{R}. \tag{8.2}$$

Let $\eta > 0$ be a small constant. By almost periodicity modulo symmetries, there exists $c(\eta) > 0$ such that

$$\int_{|\xi|\le c(\eta)N(t)} |\xi|^2 |\hat u(t,\xi)|^2\, d\xi \le \eta^2$$

for all $t \in \mathbb{R}$. On the other hand, as $u \in L_t^\infty \dot H_x^{-\varepsilon}$ for some $\varepsilon > 0$,

$$\int_{|\xi|\le c(\eta)N(t)} |\xi|^{-2\varepsilon} |\hat u(t,\xi)|^2\, d\xi \lesssim_u 1$$

for all $t \in \mathbb{R}$. Hence, by Hölder's inequality,

$$\int_{|\xi|\le c(\eta)N(t)} |\hat u(t,\xi)|^2\, d\xi \lesssim_u \eta^{\frac{2\varepsilon}{1+\varepsilon}} \quad \text{for all} \quad t \in \mathbb{R}. \tag{8.3}$$

Meanwhile, by elementary considerations and recalling that u has uniformly bounded kinetic energy,

$$\int_{|\xi|\ge c(\eta)N(t)} |\hat u(t,\xi)|^2\, d\xi \le [c(\eta)N(t)]^{-2} \int_{\mathbb{R}^d} |\xi|^2 |\hat u(t,\xi)|^2\, d\xi \lesssim_u [c(\eta)N(t)]^{-2}. \tag{8.4}$$

Collecting (8.3) and (8.4) and using Plancherel's theorem, we obtain

$$0 \le M(u) \lesssim_u c(\eta)^{-2} N(t)^{-2} + \eta^{\frac{2\varepsilon}{1+\varepsilon}}$$

for all $t \in \mathbb{R}$. As u is a low-to-high cascade, there is a sequence of times $t_n \to \infty$ so that $N(t_n) \to \infty$. As $\eta > 0$ is arbitrary, we conclude $M(u) = 0$ and hence u is identically zero. This contradicts (8.2). □

8.2. Fall of the soliton solutions. We now turn our attention to soliton-like solutions to the mass- and energy-critical NLS as described in Theorem 5.24 and 5.25 and preclude those which obey additional regularity/decay. In the defocusing case, this can be achieved using the interaction Morawetz inequality given in Proposition 7.9. We leave the precise details to the reader, noting only that the assumed regularity/decay allow one to bound the right-hand side.

In order to treat the focusing problem, we need to rely on the virial identity, which is much more closely wedded to $x = 0$. This requires us to control the motion of $x(t)$, which we do next using an argument from [**23**]. This step can be skipped over in the case of spherically symmetric initial data, since then one may take $x(t) \equiv 0$.

Lemma 8.3 (Control over $x(t)$)**.** *Suppose there is an $L_t^\infty H_x^1$ soliton-like solution to the mass-critical NLS in the sense of Theorem 5.24. Then there exists a solution u with all these properties that additionally obeys*

$$|x(t)| = o(t) \quad \textit{as} \quad t \to \infty.$$

Similarly, if u is a is a minimal kinetic energy soliton-like solution to the energy-critical NLS in the sense of Theorem 5.25 that belongs to $L_t^\infty \dot H_x^{-\varepsilon}$ for some $\varepsilon > 0$, then the same conclusion holds.

Proof. We will prove the claim for soliton-like solutions to the energy-critical NLS and leave the mass-critical case as an exercise.

We argue by contradiction. Suppose there exist $\delta > 0$ and a sequence $t_n \to \infty$ such that

$$|x(t_n)| > \delta t_n \quad \text{for all} \quad n \ge 1. \tag{8.5}$$

By spatial-translation symmetry, we may assume $x(0) = 0$.

Let $\eta > 0$ be a small constant to be chosen later. By the almost periodicity of u and Lemma 6.13, there exists $C(\eta) > 0$ such that

$$\sup_{t\in\mathbb{R}} \int_{|x-x(t)|>C(\eta)} \big(|\nabla u(t,x)|^2 + |u(t,x)|^2\big)\,dx \le \eta. \tag{8.6}$$

Define

$$T_n := \inf\{t \in [0,t_n] : |x(t)| = |x(t_n)|\} \le t_n \text{ and } R_n := C(\eta) + \sup_{t\in[0,T_n]} |x(t)|. \tag{8.7}$$

Now let ϕ be a smooth, radial function such that

$$\phi(r) = \begin{cases} 1 & \text{for } r \le 1 \\ 0 & \text{for } r \ge 2, \end{cases}$$

and define the truncated 'position'

$$X_R(t) := \int_{\mathbb{R}^d} x\phi\big(\tfrac{|x|}{R}\big)|u(t,x)|^2\,dx.$$

By hypothesis, $u \in L_t^\infty L_x^2$; together with (8.6) this implies

$$\begin{aligned} |X_{R_n}(0)| &\le \Big|\int_{|x|\le C(\eta)} x\phi\big(\tfrac{|x|}{R_n}\big)|u(0,x)|^2\,dx\Big| + \Big|\int_{|x|\ge C(\eta)} x\phi\big(\tfrac{|x|}{R_n}\big)|u(0,x)|^2\,dx\Big| \\ &\le C(\eta)M(u) + 2\eta R_n. \end{aligned}$$

On the other hand, by the triangle inequality combined with (8.6) and (8.7),

$$\begin{aligned} |X_{R_n}(T_n)| \ge{}& |x(T_n)|M(u) - |x(T_n)|\Big|\int_{\mathbb{R}^d} \Big[1-\phi\big(\tfrac{|x|}{R_n}\big)\Big]|u(T_n,x)|^2\,dx\Big| \\ &- \Big|\int_{|x-x(T_n)|\le C(\eta)} \big[x-x(T_n)\big]\phi\big(\tfrac{|x|}{R_n}\big)|u(T_n,x)|^2\,dx\Big| \\ &- \Big|\int_{|x-x(T_n)|\ge C(\eta)} \big[x-x(T_n)\big]\phi\big(\tfrac{|x|}{R_n}\big)|u(T_n,x)|^2\,dx\Big| \\ \ge{}& |x(T_n)|[M(u)-\eta] - C(\eta)M(u) - \eta[2R_n + |x(T_n)|] \\ \ge{}& |x(T_n)|[M(u)-4\eta] - 3C(\eta)M(u). \end{aligned}$$

Thus, taking $\eta > 0$ sufficiently small (depending on $M(u)$),

$$\big|X_{R_n}(T_n) - X_{R_n}(0)\big| \gtrsim_{M(u)} |x(T_n)| - C(\eta).$$

A simple computation establishes

$$\begin{aligned} \partial_t X_R(t) ={}& 2\,\mathrm{Im}\int_{\mathbb{R}^d} \phi\big(\tfrac{|x|}{R}\big)\nabla u(t,x)\overline{u(t,x)}\,dx \\ &+ 2\,\mathrm{Im}\int_{\mathbb{R}^d} \frac{x}{|x|R}\phi'\big(\tfrac{|x|}{R}\big)\,x\cdot\nabla u(t,x)\overline{u(t,x)}\,dx. \end{aligned}$$

As a minimal kinetic energy blowup solution must have zero momentum (see Corollary 2.4), using Cauchy-Schwarz and (8.6) we obtain

$$\begin{aligned} \big|\partial_t X_{R_n}(t)\big| \le{}& \Big|2\,\mathrm{Im}\int_{\mathbb{R}^d} \Big[1-\phi\big(\tfrac{|x|}{R_n}\big)\Big]\nabla u(t,x)\overline{u(t,x)}\,dx\Big| \\ &+ \Big|2\,\mathrm{Im}\int_{\mathbb{R}^d} \frac{x}{|x|R}\phi'\big(\tfrac{|x|}{R_n}\big)\,x\cdot\nabla u(t,x)\overline{u(t,x)}\,dx\Big| \end{aligned}$$

$$\leq 6\eta$$

for all $t \in [0, T_n]$.

Thus, by the Fundamental Theorem of Calculus,

$$|x(T_n)| - C(\eta) \lesssim_{M(u)} \eta T_n.$$

Recalling that $|x(T_n)| = |x(t_n)| > \delta t_n \geq \delta T_n$ and letting $n \to \infty$ we derive a contradiction. □

We are finally in a position to preclude our last enemies.

Theorem 8.4 (No solitons)**.** *There are no solutions to the mass-critical NLS that are solitons in the sense of Theorem 5.24 and that belong to* $L_t^\infty H_x^{1+\varepsilon}$ *for some* $\varepsilon > 0$*. Similarly, there are no solutions to the energy-critical NLS that are solitons in the sense of Theorem 5.25 and that belong to* $L_t^\infty \dot{H}_x^{-\varepsilon}$ *for some* $\varepsilon > 0$*.*

PROOF. We only prove the claim for the mass-critical NLS and leave the energy-critical case as exercise. Suppose for a contradiction that there existed such a solution u.

Let $\eta > 0$ be a small constant to be specified later. Then, by Definition 5.1 and Lemma 6.12 there exists $C(\eta) > 0$ such that

$$\sup_{t\in\mathbb{R}} \int_{|x-x(t)|>C(\eta)} \bigl(|u(t,x)|^2 + |\nabla u(t,x)|^2\bigr)\, dx \leq \eta. \tag{8.8}$$

Moreover, by Lemma 8.3, $|x(t)| = o(t)$ as $t \to \infty$. Thus, there exists $T_0 = T_0(\eta) \in \mathbb{R}$ such that

$$|x(t)| \leq \eta t \quad \text{for all} \quad t \geq T_0. \tag{8.9}$$

Now let ϕ be a smooth, radial function such that

$$\phi(r) = \begin{cases} r & \text{for } r \leq 1 \\ 0 & \text{for } r \geq 2, \end{cases}$$

and define

$$V_R(t) := \int_{\mathbb{R}^d} a(x)|u(t,x)|^2\, dx,$$

where $a(x) := R^2 \phi\bigl(\frac{|x|^2}{R^2}\bigr)$ for some $R > 0$.

Differentiating V_R with respect to the time variable, we find

$$\partial_t V_R(t) = 4\,\mathrm{Im} \int_{\mathbb{R}^d} \phi'\bigl(\tfrac{|x|^2}{R^2}\bigr)\overline{u(t,x)}\; x \cdot \nabla u(t,x)\, dx.$$

as in (7.6). By hypothesis $u \in L_t^\infty H_x^1$ and so we obtain

$$|\partial_t V_R(t)| \lesssim R\|\nabla u(t)\|_2 \|u(t)\|_2 \lesssim_u R \tag{8.10}$$

for all $t \in \mathbb{R}$ and $R > 0$.

Further, using (7.7) for our specific choice of a, we find

$$\begin{aligned} \partial_{tt} V_R(t) = 16E(u) &+ O\biggl(\frac{1}{R^2}\int_{|x|\geq R} |u(t,x)|^2\, dx\biggr) \\ &+ O\biggl(\int_{|x|\geq R}\Bigl[|\nabla u(t,x)|^2 + |u(t,x)|^{\frac{2(d+2)}{d}}\Bigr]\, dx\biggr). \end{aligned}$$

Recall that in the focusing case, $M(u) < M(Q)$. As a consequence, the sharp Gagliardo–Nirenberg inequality implies that the energy is a positive quantity in the focusing case as well as in the defocusing case. Indeed,

$$E(u) \gtrsim_u \int_{\mathbb{R}^d} |\nabla u(t,x)|^2\, dx > 0.$$

Thus, choosing $\eta > 0$ sufficiently small and $R := C(\eta) + \sup_{T_0 \le t \le T_1} |x(t)|$ and invoking (8.8), we obtain

$$\partial_{tt} V_R(t) \ge 8E(u) > 0. \tag{8.11}$$

Using the Fundamental Theorem of Calculus on the interval $[T_0, T_1]$ together with (8.10) and (8.11), we obtain

$$(T_1 - T_0)E(u) \lesssim_u R \lesssim_u C(\eta) + \sup_{T_0 \le t \le T_1} |x(t)|$$

for all $T_1 \ge T_0$. Invoking (8.9) and taking η sufficiently small and then T_1 sufficiently large, we derive a contradiction to $E(u) > 0$. □

Appendix A. Background material

A.1. Compactness in L^p. Recall that a family of continuous functions on a compact set $K \subset \mathbb{R}^d$ is precompact in $C^0(K)$ if and only if it is uniformly bounded and equicontinuous. This is the Arzelà–Ascoli theorem. The natural generalization to L^p spaces is due to M. Riesz [**72**] and reads as follows:

Proposition A.1. *Fix $1 \le p < \infty$. A family of functions $\mathcal{F} \subset L^p(\mathbb{R}^d)$ is precompact in this topology if and only if it obeys the following three conditions:*
(i) *There exists $A > 0$ so that $\|f\|_p \le A$ for all $f \in \mathcal{F}$.*
(ii) *For any $\varepsilon > 0$ there exists $\delta > 0$ so that $\int_{\mathbb{R}^d} |f(x) - f(x+y)|^p\, dx < \varepsilon$ for all $f \in \mathcal{F}$ and all $|y| < \delta$.*
(iii) *For any $\varepsilon > 0$ there exists R so that $\int_{|x| \ge R} |f|^p\, dx < \varepsilon$ for all $f \in \mathcal{F}$.*

Remark. By analogy to the case of continuous functions (or of measures) it is natural to refer to the three conditions as uniform boundedness, equicontinuity, and tightness, respectively.

Proof. If $\mathcal{F}$ is precompact, it may be covered by balls of radius $\frac{1}{2}\varepsilon$ around a finite collection of functions, $\{f_j\}$. As any single function obeys (i)–(iii), these properties can be extended to the whole family by approximation by an f_j.

We now turn to sufficiency. Given $\varepsilon > 0$, our job is to show that there are finitely many functions $\{f_j\}$ so that the ε-balls centered at these points cover $\mathcal{F}$. We will find these points via the usual Arzelà–Ascoli theorem, which requires us to approximate $\mathcal{F}$ by a family of continuous functions of compact support. Let $\phi : \mathbb{R}^d \to [0, \infty)$ be a smooth function supported by $\{|x| \le 1\}$ with $\phi(x) = 1$ in a neighbourhood of $x = 0$ and $\int_{\mathbb{R}^d} \phi(x)\, dx = 1$. Given $R > 0$ we define

$$f_R(x) := \phi\big(\tfrac{x}{R}\big) \int_{\mathbb{R}^d} R^d \phi\big(R(x-y)\big) f(y)\, dy$$

and write $\mathcal{F}_R := \{f_R : f \in \mathcal{F}\}$. Employing the three conditions, we see that it is possible to choose R so large that $\|f - f_R\|_p < \frac{1}{2}\varepsilon$ for all $f \in \mathcal{F}$. We also see that $\mathcal{F}_R$ is a uniformly bounded family of equicontinuous functions on the compact set $\{|x| \le R\}$. Thus, $\mathcal{F}_R$ is precompact and we may find a finite family

$\{f_j\} \subseteq C^0(\{|x| \leq R\})$ so that $\mathcal{F}_R$ is covered by the L^p-balls of radius $\frac{1}{2}\varepsilon$ around these points. By construction, the ε-balls around these points cover $\mathcal{F}$. □

In the L^2 case it is natural to replace (ii) by a condition on the Fourier transform:

Corollary A.2. *A family of functions is precompact in $L^2(\mathbb{R}^d)$ if and only if it obeys the following two conditions:*
(i) *There exists $A > 0$ so that $\|f\| \leq A$ for all $f \in \mathcal{F}$.*
(ii) *For all $\varepsilon > 0$ there exists $R > 0$ so that $\int_{|x|\geq R} |f(x)|^2\,dx + \int_{|\xi|\geq R} |\hat{f}(\xi)|^2\,d\xi < \varepsilon$ for all $f \in \mathcal{F}$.*

Proof. Necessity follows as before. Regarding the sufficiency of these conditions, we note that

$$\int_{\mathbb{R}^d} |f(x+y) - f(x)|^2\,dx \sim \int_{\mathbb{R}^d} |e^{i\xi y} - 1|^2 |\hat{f}(\xi)|^2\,d\xi,$$

which allows us to rely on the preceding proposition. □

As well as being useful in the treatment of NLS with spherically symmetric data, the following allows one to obtain tightness in the proof of Lemma A.4.

Lemma A.3 (Weighted radial Sobolev embedding). *Let $f \in H^1_x(\mathbb{R}^d)$ be spherically symmetric. Suppose $\omega : [0,\infty) \to [0,1]$ obeys $0 \leq \omega(r) \leq C\omega(\rho)$ whenever $r < \rho$. Then*

$$\big||x|^{\frac{d-1}{2}}\omega(|x|)f(x)\big|^2 \lesssim_d C^2 \|f\|_{L^2_x(\mathbb{R}^d)} \|\omega^2 \nabla f\|_{L^2_x(\mathbb{R}^d)}$$

for all $x \in \mathbb{R}^d$.

Proof. It suffices to establish the claim for spherically symmetric Schwartz functions f, which we write as functions of radius alone. Let $r \geq 0$. By the Fundamental Theorem of Calculus and the Cauchy–Schwarz inequality,

$$\begin{aligned}
r^{d-1}\omega(r)^2|f(r)|^2 &= 2r^{d-1}\omega(r)^2 \operatorname{Re}\int_r^\infty \bar{f}(\rho)f'(\rho)\,d\rho \\
&\leq 2C^2 \int_r^\infty \rho^{d-1}\omega(\rho)^2|f(\rho)|\,|f'(\rho)|\,d\rho \\
&\leq 2C^2 \Big(\int_r^\infty \rho^{d-1}|f(\rho)|^2\,d\rho\Big)^{\frac{1}{2}} \Big(\int_r^\infty \rho^{d-1}\omega(\rho)^4|f'(\rho)|^2\,d\rho\Big)^{\frac{1}{2}} \\
&\leq 2C^2 \|f\|_{L^2(\rho^{d-1}d\rho)} \|\omega^2 f'\|_{L^2(\rho^{d-1}d\rho)},
\end{aligned}$$

from which the claim follows. □

Lemma A.4 (Compactness in spherically symmetric Gagliardo–Nirenberg). *The embedding $H^1_{rad}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ is compact for $d \geq 2$ and $2 < p < \frac{2d}{d-2}$.*

Proof. Exercise. □

Our last lemma for this subsection is not strictly a compactness statement; however, it is very helpful to us in some places where we rely on weak-∗ compactness. Recall that under weak-∗ limits, the norm may jump down (i.e., the norm is weak-∗ lower semicontinuous). The question is, by how much? As we have seen in Subsection 4.2, this has a very satisfactory answer in Hilbert space (cf. (4.22)), but less so in other L^p spaces.

In our applications, regularity allows us to upgrade weak-$*$ convergence to almost everywhere convergence. The lower semicontinuity of the norm under this notion of convergence is essentially Fatou's lemma. The following quantitative version of this is due to Brézis and Lieb [**10**] (see also [**54**, Theorem 1.9]):

Lemma A.5 (Refined Fatou)**.** *Suppose* $\{f_n\} \subseteq L_x^p(\mathbb{R}^d)$ *with* $\limsup \|f_n\|_p < \infty$. *If* $f_n \to f$ *almost everywhere, then*

$$\int_{\mathbb{R}^d} \Big| |f_n|^p - |f_n - f|^p - |f|^p \Big| \, dx \to 0.$$

In particular, $\|f_n\|_p^p - \|f_n - f\|_p^p \to \|f\|_p^p$.

A.2. Littlewood–Paley theory. Let $\varphi(\xi)$ be a radial bump function supported in the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq \frac{11}{10}\}$ and equal to 1 on the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq 1\}$. For each number $N > 0$, we define the Fourier multipliers

$$\begin{aligned}
\widehat{P_{\leq N} f}(\xi) &:= \varphi(\xi/N)\hat{f}(\xi) \\
\widehat{P_{> N} f}(\xi) &:= (1 - \varphi(\xi/N))\hat{f}(\xi) \\
\widehat{P_N f}(\xi) &:= (\varphi(\xi/N) - \varphi(2\xi/N))\hat{f}(\xi)
\end{aligned}$$

and similarly $P_{<N}$ and $P_{\geq N}$. We also define

$$P_{M<\cdot\leq N} := P_{\leq N} - P_{\leq M} = \sum_{M<N'\leq N} P_{N'}$$

whenever $M < N$. We will usually use these multipliers when M and N are *dyadic numbers* (that is, of the form 2^n for some integer n); in particular, all summations over N or M are understood to be over dyadic numbers. Nevertheless, it will occasionally be convenient to allow M and N to not be a power of 2.

Like all Fourier multipliers, the Littlewood-Paley operators commute with the propagator $e^{it\Delta}$, as well as with differential operators such as $i\partial_t + \Delta$. We will use basic properties of these operators many many times, including

Lemma A.6 (Bernstein estimates)**.** *For* $1 \leq p \leq q \leq \infty$,

$$\begin{aligned}
\big\| |\nabla|^{\pm s} P_N f \big\|_{L_x^p(\mathbb{R}^d)} &\sim N^{\pm s} \|P_N f\|_{L_x^p(\mathbb{R}^d)}, \\
\|P_{\leq N} f\|_{L_x^q(\mathbb{R}^d)} &\lesssim N^{\frac{d}{p} - \frac{d}{q}} \|P_{\leq N} f\|_{L_x^p(\mathbb{R}^d)}, \\
\|P_N f\|_{L_x^q(\mathbb{R}^d)} &\lesssim N^{\frac{d}{p} - \frac{d}{q}} \|P_N f\|_{L_x^p(\mathbb{R}^d)}.
\end{aligned}$$

Lemma A.7 (Square function estimates)**.** *Given a Schwartz function* f, *let*

$$S(f)(x) := \Big(\sum \big| P_N f(x) \big|^2 \Big)^{1/2},$$

which is known as the Littlewood–Paley square function. For $1 < p < \infty$,

$$\|S(f)\|_{L_x^p} \sim \|f\|_{L_x^p}.$$

Our next estimate is a weak form of square function estimate that does not require the same amount of sparseness of the Fourier supports. We first saw this estimate as [**93**, Lemma 6.1]. While it is formulated there for rectangles, we prefer to state it for parallepipeds. It makes the proof no more involved, but reduces the amount of arithmetic required when we actually use it.

Definition A.8. A *parallelepiped* in $\mathbb{R}^d$ is a set of the form

$$R = \{Ax + c : x \in [-\tfrac{1}{2}, \tfrac{1}{2}]^d\},$$

where $A \in GL_d(\mathbb{R})$ and $c \in \mathbb{R}^d$. The variable $c = c(R)$ denotes the center of R. Given $\alpha \in (0, \infty)$, we write αR or *α-dilate of R* to refer to the parallelpiped formed from R by replacing A by αA.

Let us adopt a uniform notion of smoothed Fourier restriction operator to a parallelepiped, since we will need it in the proof below. Given $\alpha > 1$, fix a non-negative $\psi \in C_c^\infty(\mathbb{R}^d)$ with

$$\psi(x) = 1 \quad \text{for all } x \in [-\tfrac{1}{2}, \tfrac{1}{2}]^d \quad \text{and} \quad \operatorname{supp}(\psi) \subseteq [-\tfrac{\alpha}{2}, \tfrac{\alpha}{2}]^d.$$

With this fixed, we define P_R by

$$[P_R f]\hat{}(\xi) = \psi\big(A^{-1}(\xi - c)\big)\hat{f}(\xi),$$

or equivalently, by

$$P_R f = K_R * f \quad \text{where} \quad K_R(x) = |\det(A)|\, e^{ix\cdot c}\, \hat{\psi}(A^T x). \tag{A.1}$$

Here A and c are the matrix and vector used to define R. In particular, we note that

$$\int_{\mathbb{R}^d} |K_R(x)|\, dx \lesssim 1 \quad \text{uniformly in } R.$$

Lemma A.9. *Let $\{R_k\}$ be a family of parallelpipeds in $\mathbb{R}^d$ obeying*

$$\sup_\xi \sum \chi_{\alpha R_k}(\xi) \lesssim 1$$

for some $\alpha > 1$. Fix $1 \leq p \leq 2$. Then

$$\big\| \sum P_{R_k} f_k \big\|_{L^p_x(\mathbb{R}^d)}^p \lesssim \sum \big\| f_k \big\|_{L^p_x(\mathbb{R}^d)}^p$$

for any $\{f_k\} \subseteq L^p_x(\mathbb{R}^d)$.

Proof. When $p = 2$, the result follows from Plancherel's Theorem; when $p = 1$, it follows from the triangle inequality. The remaining cases can then be obtained by interpolation. □

Remark. The case $2 < p \leq \infty$ is also discussed in [**93**]; in this case, the estimate reads

$$\big\| \sum P_{R_k} f_k \big\|_{L^p_x(\mathbb{R}^d)}^{p'} \lesssim \sum \big\| f_k \big\|_{L^p_x(\mathbb{R}^d)}^{p'} \tag{A.2}$$

and the proof is essentially the same. For such p, one can actually recover the full square function estimate; see [**35, 74**].

A.3. Fractional calculus.

Lemma A.10 (Product rule, [**16**])**.** *Let $s \in (0, 1]$ and $1 < r, p_1, p_2, q_1, q_2 < \infty$ such that $\frac{1}{r} = \frac{1}{p_i} + \frac{1}{q_i}$ for $i = 1, 2$. Then,*

$$\big\| |\nabla|^s (fg) \big\|_r \lesssim \|f\|_{p_1} \big\| |\nabla|^s g \big\|_{q_1} + \big\| |\nabla|^s f \big\|_{p_2} \|g\|_{q_2}.$$

We will also need the following fractional chain rule from [**16**]. For a textbook treatment, see [**98**, §2.4].

Lemma A.11 (Fractional chain rule, [**16**])**.** *Suppose $G \in C^1(\mathbb{C})$, $s \in (0,1]$, and $1 < p, p_1, p_2 < \infty$ are such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then,*

$$\||\nabla|^s G(u)\|_p \lesssim \|G'(u)\|_{p_1} \||\nabla|^s u\|_{p_2}.$$

When the function G is no longer C^1, but merely Hölder continuous, we have the following chain rule:

Lemma A.12 (Fractional chain rule for a Hölder continuous function, [**104**])**.** *Let G be a Hölder continuous function of order $0 < \alpha < 1$. Then, for every $0 < s < \alpha$, $1 < p < \infty$, and $\frac{s}{\alpha} < \sigma < 1$ we have*

$$\big\||\nabla|^s G(u)\big\|_p \lesssim \big\||u|^{\alpha - \frac{s}{\sigma}}\big\|_{p_1} \big\||\nabla|^\sigma u\big\|^{\frac{s}{\sigma}}_{\frac{s}{\sigma} p_2}, \tag{A.3}$$

provided $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $(1 - \frac{s}{\alpha\sigma})p_1 > 1$.

The next result is formally similar to the preceding lemma; however, the proof is much simpler. It is used in the proof of Lemma 6.9.

Lemma A.13 (Nonlinear Bernstein)**.** *Let $G : \mathbb{C} \to \mathbb{C}$ be Hölder continuous of order $0 < \alpha \leq 1$. Then*

$$\|P_N G(u)\|_{L_x^{p/\alpha}(\mathbb{R}^d)} \lesssim N^{-\alpha} \|\nabla u\|^\alpha_{L^p_x(\mathbb{R}^d)}$$

for any $1 \leq p < \infty$ and $u \in \dot W^{1,p}(\mathbb{R}^d)$.

Proof. Given $h \in \mathbb{R}^d$, the Fundamental Theorem of Calculus implies

$$u(x+h) - u(x) = \int_0^1 h \cdot \nabla u(x + \theta h)\, d\theta \tag{A.4}$$

and thus,

$$\big\|G(u(x+h)) - G(u(x))\big\|_{L_x^{p/\alpha}(\mathbb{R}^d)} \lesssim |h|^\alpha \|\nabla u\|^\alpha_{L^p_x(\mathbb{R}^d)}.$$

Now let k denote the convolution kernel of the Littlewood-Paley projection P_1, so that

$$\begin{aligned}[P_N f](x) &= \int_{\mathbb{R}^d} N^d k(N(x-y)) f(y)\, dy \\ &= \int_{\mathbb{R}^d} N^d k(-Nh)[f(x+h) - f(x)]\, dh.\end{aligned}$$

Note that in obtaining the second identity, we used the fact that $\int_{\mathbb{R}^d} k(x)\, dx = 0$. Combining this with (A.4) and using the triangle inequality, we obtain

$$\begin{aligned}\|P_N G(u)\|_{L_x^{p/\alpha}(\mathbb{R}^d)} &\lesssim \|\nabla u\|^\alpha_{L^p_x(\mathbb{R}^d)} \int_{\mathbb{R}^d} |h|^\alpha N^d |k(-Nh)|\, dh \\ &\lesssim N^{-\alpha} \|\nabla u\|^\alpha_{L^p_x(\mathbb{R}^d)},\end{aligned}$$

which proves the lemma. □

Lastly, we record a particular consequence of Lemma A.12 that is used for Lemma 6.3.

Corollary A.14. *Let $0 \leq s < 1 + \frac{4}{d}$ and $F(u) = |u|^{4/d} u$. Then, on any spacetime slab $I \times \mathbb{R}^d$ we have*

$$\big\||\nabla|^s F(u)\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \lesssim \big\||\nabla|^s u\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \|u\|^{\frac{4}{d}}_{L_{t,x}^{\frac{2(d+2)}{d}}}.$$

Proof. Fix a compact interval I. Throughout the proof, all spacetime estimates will be on $I \times \mathbb{R}^d$.

For $0 < s \leq 1$, the claim is an easy consequence of Lemma A.11. It remains to address the case $1 < s < 1 + \frac{4}{d}$. We will only give details for $d \geq 5$; the main ideas carry over to lower dimensions.

Using the chain rule and the fractional product rule, we estimate as follows:

$$\begin{aligned}\big\||\nabla|^s F(u)\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} &\lesssim \big\||\nabla|^{s-1}\big(F_z(u)\nabla u + F_{\bar z}(u)\nabla \bar u\big)\big\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}\\ &\lesssim \big\||\nabla|^s u\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{4}{d}}\\ &\quad + \|\nabla u\|_{L_{t,x}^{\frac{2(d+2)}{d}}}\Big[\big\||\nabla|^{s-1}F_z(u)\big\|_{L_{t,x}^{\frac{d+2}{2}}} + \big\||\nabla|^{s-1}F_{\bar z}(u)\big\|_{L_{t,x}^{\frac{d+2}{2}}}\Big].\end{aligned}$$

The claim will follow from this, once we establish

$$\big\||\nabla|^{s-1}F_z(u)\big\|_{L_{t,x}^{\frac{d+2}{2}}} + \big\||\nabla|^{s-1}F_{\bar z}(u)\big\|_{L_{t,x}^{\frac{d+2}{2}}} \lesssim \big\||\nabla|^\sigma u\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{s-1}{\sigma}} \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{4}{d}-\frac{s-1}{\sigma}} \tag{A.5}$$

for some $\frac{d(s-1)}{4} < \sigma < 1$. Indeed, by interpolation,

$$\big\||\nabla|^\sigma u\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \lesssim \big\||\nabla|^s u\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{\sigma}{s}} \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{1-\frac{\sigma}{s}}$$

and

$$\|\nabla u\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \lesssim \big\||\nabla|^s u\big\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{1}{s}} \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{1-\frac{1}{s}}.$$

To derive (A.5), we merely observe that F_z and $F_{\bar z}$ are Hölder continuous functions of order $\frac{4}{d}$ and then apply Lemma A.12 (with $\alpha := \frac{4}{d}$ and $s := s-1$). □

A.4. A Gronwall inequality. Our last technical tool is the most elementary. It is a form of Gronwall's inequality that involves both the past and the future, 'acausal' in the terminology of [**90**]. It is used in Section 6.

Lemma A.15. *Fix $\gamma > 0$. Given $0 < \eta < \frac{1}{2}(1 - 2^{-\gamma})$ and $\{b_k\} \in \ell^\infty(\mathbb{Z}^+)$, let $x_k \in \ell^\infty(\mathbb{Z}^+)$ be a non-negative sequence obeying*

$$x_k \leq b_k + \eta \sum_{l=0}^{\infty} 2^{-\gamma|k-l|} x_l \qquad \text{for all } k \geq 0. \tag{A.6}$$

Then

$$x_k \lesssim \sum_{l=0}^{k} r^{|k-l|} b_l \qquad \text{for all } k \geq 0 \tag{A.7}$$

for some $r = r(\eta) \in (2^{-\gamma}, 1)$. Moreover, $r \downarrow 2^{-\gamma}$ as $\eta \downarrow 0$.

Proof. Our proof follows a well-travelled path. By decreasing entries in b_k we can achieve equality in (A.6); since this also reduces the righthand side of (A.7), it suffices to prove the lemma in this case. Note that since $x_k \in \ell^\infty$, b_k will remain a bounded sequence.

Let A denote the doubly infinite matrix with entries $A_{k,l} = 2^{-\gamma|k-l|}$ and let P denote the natural projection from $\ell^2(\mathbb{Z})$ onto $\ell^2(\mathbb{Z}^+)$. Our goal is to show that (A.7) holds for any solution of

$$(1 - \eta PAP^*)x = b. \tag{A.8}$$

First we observe that since

$$\|A\| = \sum_{k\in\mathbb{Z}} 2^{-\gamma|k|} = \frac{1+2^{-\gamma}}{1-2^{-\gamma}},$$

ηA is a contraction on ℓ^∞. Thus, we may write

$$x = \sum_{p=0}^{\infty} (\eta PAP^*)^p b \leq \sum_{p=0}^{\infty} P(\eta A)^p P^* b = P(1-\eta A)^{-1} P^* b,$$

where the inequality is meant entry-wise. The justification for this inequality is simply that the matrix A has non-negative entries. We will complete the proof of (A.7) by computing the entries of $(1-\eta A)^{-1}$. This is easily done via Fourier methods: Let

$$a(z) := \sum_{k\in\mathbb{Z}} 2^{-\gamma|k|} z^k = 1 + \frac{2^{-\gamma} z}{1-2^{-\gamma} z} + \frac{2^{-\gamma} z^{-1}}{1-2^{-\gamma} z^{-1}}$$

and

$$\begin{aligned} f(z) := \frac{1}{1-\eta a(z)} &= \frac{(z-2^{\gamma})(z-2^{-\gamma})}{z^2 - (2^{-\gamma} + 2^{\gamma} - \eta 2^{\gamma} + \eta 2^{-\gamma})z + 1} \\ &= 1 + \frac{(1-r2^{-\gamma})(r2^{\gamma}-1)}{(1-r^2)} \Big[1 + \frac{rz}{1-rz} + \frac{rz^{-1}}{1-rz^{-1}}\Big], \end{aligned}$$

where $r \in (0,1)$ and $1/r$ are the roots of $z^2 - (2^{-\gamma} + 2^{\gamma} - \eta 2^{\gamma} + \eta 2^{-\gamma})z + 1 = 0$. From this formula, we can immediately read off the Fourier coefficients of f, which give us the matrix elements of $(1-\eta A)^{-1}$. In particular, they are $O(r^{|k-l|})$. □

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