SPECTRA OF COMPOSITION OPERATORS ON ALGEBRAS OF ANALYTIC FUNCTIONS ON BANACH SPACES

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Abstract. Let E be a Banach space, with unit ball B_E. We study the spectrum and the essential spectrum of a composition operator on $H^\infty(B_E)$ determined by an analytic symbol with a fixed point in B_E. We relate the spectrum of the composition operator to that of the derivative of the symbol at the fixed point. We extend a theorem of Zheng to the context of analytic symbols on the open unit ball of a Hilbert space.

1. Introduction

Let E denote a complex Banach space with open unit ball B_E and let $\varphi : B_E \to B_E$ be an analytic map. In this paper, we consider composition operators C_φ defined by $C_\varphi(f) = f \circ \varphi$, acting on the uniform algebra $H^\infty(B_E)$ of bounded analytic functions on B_E. Evidently C_φ is bounded, and $||C_\varphi|| = 1 = C_\varphi(1)$. We are interested in the spectrum and the essential spectrum of C_φ. We focus on the case in which φ has a fixed point $z_0 \in B_E$. Our goal is twofold. The first is to relate the spectrum of C_φ to that of $\varphi'(z_0)$. The second is to establish an analog for higher dimensions of Zheng's theorem on the spectrum of composition operators on $H^\infty(\mathbb{D})$.

This paper is a continuation of [1], [8], and [9]. It is shown in [9] that the essential spectral radius of a composition operator on a uniform algebra is strictly less than 1 if and only if the iterates of its symbol converge in the norm of the dual to a finite number of attracting cycles. In the case at hand, the attracting cycles reduce to a single fixed point $z_0 \in B_E$. Among other things, we show under this condition that the essential spectral radius of C_φ coincides with that of $\varphi'(z_0)$.

In [20], Zheng studies composition operators C_φ on $H^\infty(\mathbb{D})$, where $\mathbb{D}$ is the open unit disk in the complex plane. Under the assumption that φ has an attracting fixed point in $\mathbb{D}$, she proves that either C_φ is power compact, in which case the essential spectral radius of C_φ is 0, or the spectrum of C_φ

2000 *Mathematics Subject Classification.* Primary 46J10. Secondary 47B38.

Key words and phrases. composition operator, spectrum, essential spectrum, interpolation.

1 Supported by Project BFM-FEDER 2003-07540 (DGI. Spain).

2 Supported partially by the Academy of Finland Project 205644 and Project BFM-FEDER 2003-07540 (DGI. Spain).

coincides with the closed unit disk. Our generalization of Zheng's theorem applies to the unit ball of $\mathbb{C}^n$ and also, subject to a compactness condition, to the unit ball of Hilbert space.

Before outlining the contents of the paper, we establish some notation. We denote by φ_n the $n$fold iterate of φ, so that $\varphi_n = \varphi \circ \varphi \circ \cdots \circ \varphi$ (n times). The spectrum and the spectral radius of an operator T are denoted respectively by $\sigma(T)$ and $r(T)$. The essential spectrum of T is denoted by $\sigma_e(T)$. It is defined to consist of all complex numbers λ such that $\lambda I - T$ is not a Fredholm operator. The essential spectral radius of T is denoted by $r_e(T)$. If $r_e(T) = 0$, then T is said to be a Riesz operator. For background information on the essential spectrum and on Fredholm operators, see [17]. For background information on analytic functions on Banach spaces and the associated tensor product spaces, see [6], [10], or [16]. References on composition operators on uniform algebras are [14], [15], [11], and [12].

The paper is organized as follows. After some preliminary lemmas on lower triangular matrices in Section 2, we treat in Section 3 the lower triangular representation of C_φ corresponding to the Taylor series expansion of functions in $H^\infty(B_E)$ at the fixed point z_0. The results are applied in Section 4 to the case where $r_e(C_\varphi) < 1$. There we connect the spectrum and essential spectrum of C_φ to that of $\varphi'(z_0)$. In particular, we show that if C_φ is a Riesz operator, then $\varphi'(z_0)$ is a Riesz operator, and we describe the spectrum of C_φ in terms of that of $\varphi'(z_0)$. This generalizes the corresponding result obtained in [1] for compact composition operators.

Sections 5 and 6 contain preparatory material for the generalization of Zheng's theorem. In Section 5 we observe that the interpolation operators constructed by B. Berndtsson [2] for the unit ball of $\mathbb{C}^n$ yield an interpolation theorem for sequences in the unit ball B_H of Hilbert space H that tend exponentially to the boundary. In Section 6 we establish a Julia-type estimate for analytic self-maps of B_H as the variable tends to the boundary through an approach region that clusters on a compact subset of the unit sphere. These results are combined in Section 7 to establish the generalization of Zheng's theorem.

2. Spectra of Lower Triangular Operators

We begin with the following elementary fact.

Lemma 2.1. *A lower triangular square matrix with entries in a unital ring is invertible and has a lower triangular inverse if and only if the diagonal entries of the matrix are invertible.*

Proof. Backsolve. □

We will be interested in operators on a direct sum $X = X_1 \oplus \cdots \oplus X_n$ of Banach spaces. Such an operator S leaves invariant each direct subsum

$X_k \oplus \cdots \oplus X_n$, $2 \le k \le n$, if and only if S has a lower triangular matrix representation

$$S = \begin{pmatrix} S_{11} & 0 & 0 & \dots & 0 \\ S_{21} & S_{22} & 0 & \dots & 0 \\ \vdots & & & & \\ \dots & \dots & \dots & S_{n-1,n-1} & 0 \\ S_{n1} & S_{n2} & \dots & S_{n,n-1} & S_{nn} \end{pmatrix}, \tag{2.1}$$

where $S_{jk} : X_j \to X_k$. From the preceding lemma, applied to the lower triangular matrix operator $\lambda I - S$, we see that if each of the diagonal entries $\lambda I - S_{jj}$ is an invertible operator on its space X_j, then $\lambda I - S$ is invertible. Similarly, if we apply the lemma to the quotient ring of operators modulo compact operators, we see that if each of the diagonal entries $\lambda I - S_{jj}$ is a Fredholm operator on its space X_j, then $\lambda I - S$ is a Fredholm operator on X. This yields the following lemma.

Lemma 2.2. *Let $X = X_1 \oplus \cdots \oplus X_n$ be a direct sum of Banach spaces, and let S be an operator on X with lower triangular matrix representation* (2.1)*. Then*

$$\sigma(S) \subseteq \sigma(S_{11}) \cup \cdots \cup \sigma(S_{nn}), \tag{2.2}$$

and

$$\sigma_e(S) \subseteq \sigma_e(S_{11}) \cup \cdots \cup \sigma_e(S_{nn}), \tag{2.3}$$

where $\sigma(S_{jj})$ and $\sigma_e(S_{jj})$ are respectively the spectrum and essential spectrum of S_{jj} operating on X_j.

It can occur that the inclusions in (2.2) and (2.3) are strict. Nevertheless we have the following.

Lemma 2.3. *Let $X = X_1 \oplus \cdots \oplus X_n$ be a direct sum of Banach spaces, and let S be an operator on X with lower triangular matrix representation* (2.1)*. Let Ω be the unbounded component of the complement of $\sigma_e(S_{11}) \cup \cdots \cup \sigma_e(S_{n-1,n-1})$ in the complex plane $\mathbb{C}$. Then*

$$\sigma(S) \cap \Omega = (\sigma(S_{11}) \cup \cdots \cup \sigma(S_{nn})) \cap \Omega, \tag{2.4}$$

$$\sigma_e(S) \cap \Omega = \sigma_e(S_{nn}) \cap \Omega. \tag{2.5}$$

Further, $\partial\Omega \subseteq \sigma_e(S)$.

Proof. Let $\lambda_0 \in \Omega \backslash \sigma(S)$. To establish (2.4), we must show that $\lambda_0 \notin \sigma(S_{11}) \cup \cdots \cup \sigma(S_{nn})$, that is, we must show that each $\lambda_0 I - S_{jj}$ is invertible. We break the argument into two cases.

Suppose first that $\lambda_0 \notin \sigma(S_{11}) \cup \cdots \cup \sigma(S_{n-1,n-1})$. We must show that $\lambda_0 \notin \sigma(S_{nn})$. For this, let U be the lower triangular $(n-1) \times (n-1)$ matrix obtained by striking out the last column and the bottom row of S, so that

$$S = \begin{pmatrix} U & 0 \\ V & S_{nn} \end{pmatrix}.$$

By Lemma 2.1, $\lambda_0 I - U$ is invertible and its inverse is lower triangular. The inverse of $\lambda_0 I - S$ then has the form

$$(\lambda_0 I - S)^{-1} = \begin{pmatrix} (\lambda_0 I - U)^{-1} & R \\ T & W \end{pmatrix}.$$

Multiplying by $\lambda_0 I - S$ on the left, we find that the column vector R satisfies $(\lambda_0 I - U)R = 0$. Since $\lambda_0 I - U$ is invertible, $R = 0$. Consequently $\lambda_0 I - S$ has a lower triangular inverse, and $W = (\lambda_0 I - S_{nn})^{-1}$. In particular, $\lambda_0 \notin \sigma(S_{nn})$, as required.

For the remaining case, suppose that $\lambda_0 \in \sigma(S_{11}) \cup \cdots \cup \sigma(S_{n-1,n-1})$. We will show that this leads to a contradiction. For this, we apply the first part of the proof to λ's in a punctured neighborhood of λ_0. Since $\sigma(S_{11}) \cup \cdots \cup \sigma(S_{n-1,n-1})$ meets Ω in a discrete subset, by what we have shown there is a punctured neighborhood of λ_0 on which $\lambda I - S$ has a lower triangular inverse. Letting $\lambda \to \lambda_0$, we see that the inverse of $\lambda_0 I - S$ is also lower triangular. Hence $\lambda_0 \notin \sigma(S_{jj})$ for $1 \leq j \leq n$. This is a contradiction, and we conclude that (2.4) holds.

The same argument as in the first case above, applied to the quotient algebra of operators modulo compact operators, shows that if $\lambda \in \Omega$ and $\lambda \notin \sigma_e(S)$, then $\lambda \notin \sigma_e(S_{nn})$. Hence (2.5) holds, and further the inverse of $\lambda I - S$ modulo the compacts has a lower triangular matrix representation when $\lambda \in \Omega \backslash \sigma_e(S)$.

Now let $\lambda_0 \in \partial\Omega$, and suppose that $\lambda_0 \notin \sigma_e(S)$. By the preceding remark, there are $\lambda \in \Omega$ near λ_0 for which the inverse of $\lambda I - S$ modulo the compacts has a lower triangular matrix representation. Letting $\lambda \to \lambda_0$, we see that the inverse of $\lambda_0 I - S$ modulo the compacts has a lower triangular matrix representation. Consequently each of the diagonal entries $\lambda_0 I - S_{jj}$ is invertible modulo the compacts, and $\lambda_0 \in \Omega$. This contradiction shows that $\lambda_0 \in \sigma_e(S)$, and the proof is complete. □

Corollary 2.4. *Let $X = X_1 \oplus \cdots \oplus X_n$ be a direct sum of Banach spaces, and let S be an operator on X with a lower triangular matrix representation* (2.1). *If X is infinite dimensional, and the operators $S_{11}, \ldots, S_{n-1,n-1}$ are Riesz operators, then $\sigma(S) = \sigma(S_{11}) \cup \cdots \cup \sigma(S_{nn})$ and $\sigma_e(S) = \{0\} \cup \sigma_e(S_{nn})$.*

Proof. Apply the lemma, with $\Omega = \mathbb{C} \backslash \{0\}$. □

3. Composition Operators on $H^\infty(B_E)$

Let E be a Banach space, with open unit ball B_E. We will use the pseudohyperbolic metric $\rho(z, w)$ on B_E determined by the uniform algebra $H^\infty(B_E)$ (see [8], [9]). This is defined for $z, w \in B_E$ by

$$\rho(z, w) = \sup\{\rho_{\mathbb{D}}(f(z), f(w)) : f \in H^\infty(B_E), ||f|| < 1\}, \tag{3.1}$$

where $\rho_{\mathbb{D}}(\zeta, \xi) = |\zeta - \xi|/|1 - \bar{\xi}\zeta|$ is the pseudohyperbolic metric on the open unit disk $\mathbb{D}$.

If B_E has a transitive group of automorphisms, then $\rho(z,w)$ is the norm of the evaluation functional at $\phi(z)$ on the subspace of functions in $H^\infty(B_E)$ that vanish at 0, where ϕ is any automorphism of B_E that maps w to 0. In the special case that E is a Hilbert space H, the pseudohyperbolic metric on B_H is determined by the explicit formula (see [5], pp. 194-195),

$$\rho(z,w)^2 = 1 - \frac{(1-||z||^2)(1-||w||^2)}{|1-\langle z,w\rangle|^2}, \qquad z,w \in B_H. \tag{3.2}$$

Now suppose that the analytic map $\varphi : B_E \to B_E$ has a fixed point $z_0 \in B_E$, that is, $\varphi(z_0) = z_0$. Let $\mathcal{P}_k$ be the space of analytic functions $f(z)$ on E such that $f(z) = P(z - z_0)$ for some homogeneous polynomial P on E of degree k. The Taylor series expansions of functions in $H^\infty(B_E)$ at z_0 yield a direct sum decomposition of $H^\infty(B_E)$,

$$H^\infty(B_E) = \mathcal{P}_0 \oplus \mathcal{P}_1 \oplus \cdots \oplus \mathcal{P}_{m-1} \oplus \mathcal{R}_m, \tag{3.3}$$

where the remainder space $\mathcal{R}_m$ consists of the functions $h \in H^\infty(B_E)$ such that $|h(z)|/||z - z_0||^m$ is bounded for z near z_0. In other words, a function belongs to $\mathcal{R}_m$ if and only if the terms of its Taylor expansion at z_0 vanish through order $m - 1$. The maximum principle yields the estimates

$$|f(z)| \le c_m ||f||\, ||z - z_0||^m, \qquad f \in \mathcal{R}_m,\ ||z - z_0|| \le 1 - ||z_0||, \tag{3.4}$$

where $c_m = 1/(1 - ||z_0||)^m$ and the norm of f is the supremum norm over B_E (see [16], 7.19). The same estimate holds for $f \in \mathcal{P}_k$, with c_m replaced by c_k.

Now note that the order of vanishing of $f \circ \varphi$ at z_0 is at least as great as the order of vanishing of f at z_0, on account of (3.4). Consequently $\mathcal{R}_m$ and each of the spaces $\mathcal{P}_k \oplus \cdots \oplus \mathcal{P}_{m-1} \oplus \mathcal{R}_m$ are invariant under C_φ. Thus C_φ has a lower triangular representation with respect to the sum decomposition (3.3), which we denote by

$$C_\varphi = \begin{pmatrix} C_{00} & 0 & 0 & \dots & 0 \\ C_{10} & C_{11} & \dots & \dots & 0 \\ \vdots & & & & \\ \dots & \dots & \dots & C_{m-1,m-1} & 0 \\ C_{m0} & C_{m1} & \dots & C_{m,m-1} & Q_m \end{pmatrix}. \tag{3.5}$$

Note that the operator Q_m is simply the restriction of C_φ to $\mathcal{R}_m$. We wish to identify the other diagonal terms $C_{kk} : \mathcal{P}_k \to \mathcal{P}_k$.

Since $\mathcal{P}_0$ consists of only the constants, and $C_\varphi f = f$ for $f \in \mathcal{P}_0$, we see that $C_{00} = I$ and $C_{k0} = 0$ for $1 \le k \le n$. The space $\mathcal{P}_1$ is isomorphic to E^*, and it is straightforward to check that C_{11} corresponds to the dual of $\varphi'(z_0)$ under the isomorphism. In fact, the diagonal terms C_{kk}, $1 \le k \le m-1$, can all be expressed in terms of operators induced by φ on certain tensor product spaces. We sketch how this is done. (See [6], [10], [16].)

Since our considerations are local, we consider an analytic map ψ of a neighborhood of $0 \in E$ into E such that $\psi(0) = 0$. The map ψ induces a map

$\hat{\otimes}^k\psi'(0)$ of the projective tensor product $\hat{\otimes}^k E$ determined by $x_1\otimes\cdots\otimes x_k \to (\psi'(0)x_1)\otimes\cdots\otimes(\psi'(0)x_k)$. The dual of the projective tensor product is the space $\mathcal{L}({}^kE)$ of continuous k-linear functionals on E, and the dual of $\hat{\otimes}^k\psi'(0)$ is the operator on $\mathcal{L}({}^kE)$ defined by

$$((\hat{\otimes}^k\psi'(0))^*L)(x_1,\ldots,x_m) = L(\psi'(0)x_1,\ldots,\psi'(0)x_m).$$

Clearly $||\hat{\otimes}^k\psi'(0)|| \le ||\psi'(0)||^m$, and by choosing $x_1 = \cdots = x_k$ and L appropriately we easily obtain $||\hat{\otimes}^k\psi'(0)|| = ||\psi'(0)||^m$.

The symmetric tensors form a closed subspace $\hat{\otimes}^k_s E$ of $\hat{\otimes}^k E$, whose dual is the space $\mathcal{L}_s({}^kE)$ of symmetric k-linear functionals on E. The usual symmetrization operation yields norm-one projections of $\hat{\otimes}^k E$ onto $\hat{\otimes}^k_s E$ and of $\mathcal{L}({}^kE)$ onto $\mathcal{L}_s({}^kE)$, which are dual. Symmetrization commutes with the operator $\hat{\otimes}^k\psi'(0)$ and its dual, so that in particular $\hat{\otimes}^k\psi'(0)$ leaves $\hat{\otimes}^k_s E$ invariant. We denote by $\hat{\otimes}^k_s\psi'(0)$ the restriction of $\hat{\otimes}^k\psi'(0)$ to $\hat{\otimes}^k_s E$. The dual operator $(\hat{\otimes}^k_s\psi'(0))^*$ is then the restriction of $(\hat{\otimes}^k\psi'(0))^*$ to $\mathcal{L}_s({}^kE)$. One checks that

$$||\hat{\otimes}^k_s\psi'(0)|| = ||\psi'(0)||^k, \qquad k \ge 1.$$

Lemma 3.1. *With the notation introduced above, we have*

$$\sigma(\hat{\otimes}^k\psi'(0)) = \{\lambda_1\cdots\lambda_k : \lambda_j \in \sigma(\psi'(0)),\, 1 \le j \le k\}, \tag{3.6}$$

$$\sigma_e(\hat{\otimes}^k\psi'(0)) = \{\lambda_1\cdots\lambda_k : \lambda_1 \in \sigma_e(\psi'(0)),\, \lambda_j \in \sigma(\psi'(0)),\, 2 \le j \le k\}. \tag{3.7}$$

Further,

$$\sigma(\hat{\otimes}^k_s\psi'(0)) \subseteq \sigma(\hat{\otimes}^k\psi'(0)), \tag{3.8}$$

$$\sigma_e(\hat{\otimes}^k_s\psi'(0)) \subseteq \sigma_e(\hat{\otimes}^k\psi'(0)). \tag{3.9}$$

Proof. If S and T are bounded operators on a Banach space X, then the projective tensor product operator $S\hat{\otimes}T$ has spectrum $\sigma(S\hat{\otimes}T) = \sigma(S)\sigma(T)$ (see e.g. [18]) and essential spectrum $\sigma_e(S\hat{\otimes}T) = \sigma_e(S)\sigma(T) \cup \sigma(S)\sigma_e(T)$ (see e.g. [19] or [7]). By induction, we obtain (3.6) and (3.7). The inclusions (3.8) and (3.9) follow from the fact that $\hat{\otimes}^k_s\psi'(0)$ commutes with the symmetrization projections. □

Let $\mathcal{P}({}^kE)$ denote the space of all analytic functions on E that are k-homogeneous. The functions in $\mathcal{P}({}^kE)$ are precisely the restrictions to the diagonal of symmetric k-linear functionals on E, and the restriction operator is an isomorphism (though not isometry) of $\mathcal{L}_s({}^kE)$ and $\mathcal{P}({}^kE)$. One checks that under this isomorphism, the operator $(\otimes^k_s\psi'(0))^*$ corresponds to the operator C_{kk} in the lower triangular representation of C_ψ determined by the decomposition

$$H^\infty(B_E) = \mathcal{P}({}^0E) \oplus \mathcal{P}({}^1E) \oplus \cdots \oplus \mathcal{P}({}^{m-1}E) \oplus \cdots .$$

Now we return to our analytic map φ with fixed point z_0. The translation $z \to z - z_0$ establishes an isomorphism (though not an isometry unless $z_0 = 0$) of $\mathcal{P}_k$ and $\mathcal{P}({}^kE)$. Applying the preceding discussion to $\psi(z) = \varphi(z_0 + z) - z_0$, we obtain the following.

Lemma 3.2. *Let E be a Banach space, and let $\varphi : B_E \to B_E$ be an analytic map with a fixed point $z_0 \in B_E$. Let $m \geq 1$, and let (3.5) be the associated lower triangular matrix representation of C_φ. Then for $1 \leq k \leq m-1$, C_{kk} is similar to the operator $(\hat{\otimes}_s^k \varphi'(z_0))^*$ operating on symmetric k-linear functionals on E. There is a constant $a_k > 0$, depending only on k and z_0, such that*

$$||C_{kk}|| \leq a_k ||\varphi'(z_0)||^k.$$

For $1 \leq k \leq m-1$

$$\sigma(C_{kk}) \subseteq \{\lambda_1 \cdots \lambda_k : \lambda_j \in \sigma(\varphi'(z_0)),\ 1 \leq j \leq k\}, \tag{3.10}$$

$$\sigma_e(C_{kk}) \subseteq \{\lambda_1 \cdots \lambda_k : \lambda_1 \in \sigma_e(\varphi'(z_0)),\ \lambda_j \in \sigma(\varphi'(z_0)),\ 2 \leq j \leq k\}. \tag{3.11}$$

Further, $r(C_{kk}) \leq r(\varphi'(z_0))^k$ and $r_e(C_{kk}) \leq r_e(\varphi'(z_0)) r(\varphi'(z_0))^{k-1}$.

We remark that the estimate for the spectral radius of C_{kk} can also be obtained directly by applying the norm estimates to the iterates C_φ^n and observing that the lower triangular matrix representation corresponding to the nth iterate of φ has diagonal entries C_{kk}^n that correspond to $\varphi'(z_0)^n$. Consequently

$$||C_{kk}^n|| \leq a_k ||\varphi'(z_0)^n||^k \leq a_k ||\varphi'(z_0)^n||^k.$$

Taking nth roots and letting $n \to \infty$, we obtain $r(C_{kk}) \leq r(\varphi'(z_0))^k$.

Now we combine Lemmas 2.2, 2.3, and 3.2, to obtain the main result of this section.

Theorem 3.3. *Let E be a Banach space, and let $\varphi : B_E \to B_E$ be an analytic map with a fixed point $z_0 \in B_E$. Let $m \geq 1$, and let (3.5) be the associated lower triangular matrix representation of C_φ. Let*

$$\Lambda = \{1\} \cup \{\lambda_1 \lambda_2 \cdots \lambda_k : \lambda_j \in \sigma(\varphi'(z_0)),\ 1 \leq j \leq k,\ k \geq 1\}.$$

Then

$$\sigma(C_\varphi) \subseteq \Lambda \cup \sigma(Q_m) \tag{3.12}$$

and

$$r_e(C_\varphi) = \max\{r_e(\varphi'(z_0)), r_e(Q_m)\}. \tag{3.13}$$

If $|\lambda| > r_e(C_\varphi)$, then $\lambda \in \sigma(C_\varphi)$ if and only if $\lambda \in \Lambda \cup \sigma(Q_m)$. If $|\lambda| > r_e(\varphi'(z_0))$, then $\lambda \in \sigma_e(C_\varphi)$ if and only if $\lambda \in \sigma_e(Q_m)$.

Proof. We take the S of (2.1) to be the C_φ of (3.5). Then the inclusion (3.12) follows from Lemma 3.2 and the inclusion (2.2) of Lemma 2.2. Since $r_e(C_{kk}) \leq r_e(C_{11}) = r_e(\varphi'(z_0))$ for $2 \leq k \leq m-1$, the set Ω of Lemma 2.3 includes all $\lambda \in \mathbb{C}$ such that $|\lambda| > r_e(\varphi'(z_0))$. Further, any point $\lambda_0 \in \sigma_e(\varphi'(z_0))$ satisfying $|\lambda_0| = r_e(\varphi'(z_0))$ belongs to $\partial\Omega$, hence by Lemma 2.3

to $\sigma_e(C_\varphi)$. By (2.5), $\sigma_e(C_\varphi) \cap \Omega = \sigma_e(Q_m) \cap \Omega$. Thus the final assertion of the theorem holds, and this implies (3.13).

It remains to establish the penultimate assertion of the theorem, and for this it suffices to show that any $\lambda \in \Lambda$ satisfying $|\lambda| > r_e(C_\varphi)$ belongs to $\sigma(C_\varphi)$.

So suppose $\lambda = \lambda_1 \cdots \lambda_k$ satisfies $|\lambda| > r_e(C_\varphi)$, and $\lambda_j \in \sigma(\varphi'(z_0))$ for $1 \le j \le k$. Then $|\lambda_j| > r_e(C_\varphi)$. By (2.4) of Lemma 2.3, $\lambda_j \in \sigma(C_\varphi)$. Consequently λ_j is an eigenvalue of C_φ, and there is a function $f_j \in H^\infty(B_E)$, f_j not the zero function, such that $f_j \circ \varphi = \lambda_j f_j$. Then $f = f_1 \cdots f_k$ satisfies $f \circ \varphi = \lambda f$, so f is a nonzero eigenfunction of C_φ with eigenvalue λ, and $\lambda \in \sigma(C_\varphi)$. □

We do not know whether it is always the case in Theorem 3.3 that every $\lambda \in \Lambda$ is an eigenvalue of C_φ, nor whether equality holds in (3.12).

4. The Case $r_e(C_\varphi) < 1$

In this section, we focus on composition operators C_φ on $H^\infty(B_E)$ for which $r_e(C_\varphi) < 1$. By Corollary 2.4 of [9], this occurs if and only if there are $\varrho < 1$ and $n \ge 1$ such that $\varphi_n(B_E) \subseteq \varrho B_E$. In this case the iterates φ_n of φ converge uniformly on B_E to a unique fixed point $z_0 \in B_E$.

On account of the eigenvalue $\lambda = 1$, the spectral radius of C_φ is $r(C_\varphi) = 1$. In the presence of an attracting fixed point z_0, it is natural to split off the eigenspace of this eigenvalue and to consider the restriction of C_φ to the subspace $\mathcal{R}_1$ of functions that vanish at the fixed point. We begin with a formula for the spectral radius of the restricted operator.

Theorem 4.1. *Let E be a Banach space, and let $\varphi : B_E \to B_E$ be an analytic map such that $r_e(C_\varphi) < 1$. Let $z_0 \in B_E$ be the fixed point of φ, and let $\mathcal{R}_1$ be the subspace of functions in $H^\infty(B_E)$ that vanish at z_0. Then the spectral radius of the restriction of C_φ to $\mathcal{R}_1$ is given by*

$$r(C_\varphi|_{\mathcal{R}_1}) = \lim_{n\to\infty} \|\varphi_n - z_0\|^{1/n}. \tag{4.1}$$

Further, $r(C_\varphi|_{\mathcal{R}_1}) < 1$.

Proof. Let $\varepsilon > 0$. If z is near z_0 and $f \in \mathcal{R}_1$, we have

$$\begin{aligned} |f(\varphi(z))| &\le (||f'(z_0)|| + \varepsilon ||f||)(||\varphi'(z_0)|| + \varepsilon)||z - z_0|| \\ &\le c||f||(||\varphi'(z_0)|| + \varepsilon)||z - z_0||. \end{aligned}$$

Since $\varphi_n(z) \to z_0$ uniformly for $z \in B_E$, we obtain for large n the estimate

$$|f(\varphi_n(z))| \le c||f||(||\varphi'(z_0)|| + \varepsilon)||\varphi_n(z) - z_0||, \qquad z \in B_E.$$

Hence for large n we have

$$||C_\varphi^n|_{\mathcal{R}_1}|| \le c(||\varphi'(z_0)|| + \varepsilon)||\varphi_n(z) - z_0||.$$

Taking nth roots and letting $n \to \infty$, we see that

$$r(C_\varphi|_{\mathcal{R}_1}) \le \liminf \|\varphi_n - z_0\|^{1/n}. \tag{4.2}$$

For the reverse inequality, let $\varepsilon > 0$ be small again, and choose $z_n \in B_E$ such that $||\varphi_n(z_n) - z_0|| > \|\varphi_n - z_0\| - \varepsilon$. Choose $g \in E^*$ such that $||g|| = 1$ and $|g(\varphi_n(z_n) - z_0)| > \|\varphi_n - z_0\| - \varepsilon$. Then $g - g(z_0) \in \mathcal{R}_1$ satisfies $||g - g(z_0)|| \leq 1 + ||z_0||$, and

$$||C_\varphi^n|_{\mathcal{R}_1}|| \geq \frac{||C_\varphi^n(g - g(z_0))||}{||g - g(z_0)||} \geq \frac{|g(\varphi_n(z_n)) - g(z_0)|}{1 + ||z_0||} \geq \frac{\|\varphi_n - z_0\| - \varepsilon}{1 + ||z_0||}.$$

Letting $\varepsilon \to 0$, taking nth roots, and letting $n \to \infty$, we see that

$$r(C_\varphi|_{\mathcal{R}_1}) \geq \limsup \|\varphi_n - z_0\|^{1/n}. \tag{4.3}$$

From (4.2) and (4.3), we see that $\|\varphi_n - z_0\|^{1/n}$ has a limit as $n \to \infty$, and the limit is given by (4.1).

For the final statement, note that the linear span of the eigenspaces of C_φ corresponding to the eigenvalues of unit modulus form a finite dimensional subalgebra of $H^\infty(B_E)$. Since any finite dimensional uniform algebra is the linear span of its nonzero minimal idempotents, and since the only nonzero idempotent in $H^\infty(B_E)$ is the constant function 1, in fact C_φ has only one eigenvalue on the unit circle, a simple eigenvalue at $\lambda = 1$ with eigenfunction 1. It follows that $r(C_\varphi|_{\mathcal{R}_1}) < 1$. □

Corollary 4.2. *Let E be a Banach space, and let $\varphi : B_E \to B_E$ be an analytic map such that $r_e(C_\varphi) < 1$. Let $z_0 \in B_E$ be the fixed point of φ, and let $\mathcal{R}_1$ be the subspace of functions in $H^\infty(B_E)$ that vanish at z_0. Let $m \geq 1$, and let* (3.5) *be the associated matrix representation of C_φ. Then*

$$r(Q_m) \leq r(C_\varphi|_{\mathcal{R}_1})^m. \tag{4.4}$$

Proof. Applying (3.4) with z replaced by $\varphi_n(z)$ and n big enough, we obtain $|(C_\varphi^n f)(z)| \leq c_m||\varphi_n(z) - z_0||^m||f||$ for $f \in \mathcal{R}_m$ and $z \in B_E$. Consequently $||C_\varphi^n|| \leq c_m||\varphi_n - z_0||^m$. Since Q_m^n is the restriction of C_φ^n to $\mathcal{R}_m$, we have $||Q_m^n|| \leq c_m||\varphi_n - z_0||^m$. Taking nth roots and letting $n \to \infty$, we obtain (4.4). □

The relation between the spectral radii and the essential spectral radii of C_φ and φ' is given in the following theorem.

Theorem 4.3. *Let E be a Banach space, and let $\varphi : B_E \to B_E$ be an analytic map such that $r_e(C_\varphi) < 1$. Let $z_0 \in B_E$ be the fixed point of φ, and let $\mathcal{R}_1$ be the subspace of functions in $H^\infty(B_E)$ that vanish at z_0. Then*

$$r(C_\varphi|_{\mathcal{R}_1}) = r(\varphi'(z_0)), \tag{4.5}$$

and

$$r_e(C_\varphi) = r_e(\varphi'(z_0)). \tag{4.6}$$

If $|\lambda| > r_e(C_\varphi)$, then $\lambda \in \sigma(C_\varphi)$ if and only if $\lambda \in \Lambda$, where Λ is as in Theorem 3.3.

Proof. By Corollary 4.2, we can choose m so large that $r(Q_m)$ is arbitrarily small. Hence (4.6) follows from (3.13). The final assertion of the theorem then follows from the corresponding assertion of Theorem 3.3. The identity (4.5) follows from this and the definition of Λ. □

We wish to specialize the results of this section to Riesz operators. For this, we need the following lemma.

Lemma 4.4. *Let E be a Banach space, let $\varphi : B_E \to B_E$ be an analytic map, and let $w \in B_E$. Then $||\varphi_n - w||^{1/n}$ has a limit as $n \to \infty$. Either $||\varphi_n - w||^{1/n} \to 1$ as $n \to \infty$, or else $r_e(C_\varphi) < 1$ and $w = z_0$ is the fixed point of φ, in which case the limit is given by Theorem* 4.1.

Proof. Since $||\varphi_n - w|| \leq 2$, $\limsup ||\varphi_n - w||^{1/n} \leq 1$. If $r_e(C_\varphi) = 1$, then for each n the image $\varphi_n(B_E)$ is not contained in any proper subball centered at 0, so $||\varphi_n - w||$ is bounded away from 0, and $||\varphi_n - w||^{1/n} \to 1$. If $r_e(C_\varphi) < 1$ and w is not the fixed point of φ, then again $||\varphi_n - w||$ is bounded away from 0, and $||\varphi_n - w||^{1/n} \to 1$. □

The following is now an immediate consequence of Theorem 4.3 and Lemma 4.4.

Theorem 4.5. *Let E be a Banach space, and let $\varphi : B_E \to B_E$ be an analytic map. If φ has a fixed point $z_0 \in B_E$ such that $\varphi'(z_0)$ is a Riesz operator and*

$$\lim_{n\to\infty} ||\varphi_n - z_0||^{1/n} < 1, \tag{4.7}$$

then C_φ is a Riesz operator. Conversely, if C_φ is a Riesz operator, then φ has an attracting fixed point $z_0 \in B_E$, $\varphi'(z_0)$ is a Riesz operator, and $\lim ||\varphi_n - z_0||^{1/n} < 1$. *In this case, the spectrum of C_φ consists of $\lambda = 0$ and $\lambda = 1$, together with all possible products $\lambda = \lambda_1 \cdots \lambda_k$, where $k \geq 1$ and the λ_j's are eigenvalues of $\varphi'(z_0)$.*

Example 4.6. As a simple example, consider the case where $\varphi = T$ is a linear operator on E satisfying $||T|| < 1$. The fixed point of φ is at $z_0 = 0$, and $\varphi'(0) = T$. The matrix representation of C_φ is diagonal, and C_{kk} is similar to $(\hat{\otimes}_s^k T)^*$. In this case, $r(C_\varphi|_{\mathcal{R}_1}) = r(T)$ and $r_e(C_\varphi) = r_e(T)$. By choosing T to be a quasinilpotent operator that is not power compact, we obtain in this way a composition operator C_φ that is not power compact but whose restriction to the subspace $\mathcal{R}_1$ (of codimension 1) is quasinilpotent.

5. Interpolating Sequences in the Unit Ball of Hilbert Space

Let H be a Hilbert space, with unit ball B_H. We are interested in sequences $\{z_k\}_{k=1}^\infty$ in B_H that tend exponentially to the boundary of B_H in the sense that there is a constant $b < 1$ such that

$$\frac{1 - \|z_{k+1}\|}{1 - \|z_k\|} \leq b, \qquad k \geq 1. \tag{5.1}$$

We will need the following fact.

Theorem 5.1. *Let $0 < r < 1$ and let $0 < b < 1$. Let $\{z_k\}_{k=1}^{\infty}$ be a sequence in B_H such that $\|z_1\| \geq r$ and such that* (5.1) *holds. Then $\{z_k\}_{k=1}^{\infty}$ is an interpolating sequence, with interpolation constant M that depends only on b and r.*

The proof boils down to a simple application of the work of Berndtsson [2]. We explain how it comes about.

According to Carleson's theorem, a necessary and sufficient condition for a sequence of points z_k in the open unit disk $\mathbb{D}$ to be an interpolating sequence for $H^\infty(\mathbb{D})$ is that there exists $\delta > 0$ such that

$$\prod_{j \neq k} \rho(z_j, z_k) \geq \delta, \qquad 1 \leq k < \infty, \tag{5.2}$$

where ρ is here the pseudohyperbolic metric $\rho_{\mathbb{D}}$ of the open unit disk. Further, the interpolation constant of the sequence depends only on δ. Berndtsson was able to extend to several variables a construction for the interpolating functions due to P. Jones and show that the condition (5.2), where ρ is now the pseudohyperbolic metric of the open unit ball B_n of $\mathbb{C}^n$, is a sufficient condition for a sequence of points in B_n to be an interpolating sequence for $H^\infty(B_n)$. As pointed out in [3], Berndtsson's interpolation constant depends only on δ and not on the dimension n. By interpolating on finite subsets of the sequence with uniform bounds and applying a normal families argument, we can pass to a limit as $n \to \infty$. We then have the following version of Berndtsson's theorem, where ρ is now the pseudohyperbolic metric on B_H.

Theorem 5.2. *Let H be a Hilbert space, and let $\delta > 0$. There is a constant $M > 0$ such that any sequence $\{z_k\}$ in B_H satisfying* (5.2) *is an interpolating sequence for $H^\infty(B_H)$ with interpolation constant $\leq M$.*

The examples given by Berndtsson show that the condition (5.2) is far from being a necessary condition for interpolation. In fact, the explicit form of the metric $\rho(z, w)$ given in (3.2) shows that any sequence $\{z_k\}$ satisfying (5.2) satisfies $\sum(1 - \|z_k\|) < \infty$. An interpolating sequence for $H^\infty(B_H)$ that does not satisfy this condition is given by $\{z_k = w_k/2\}$, where $\{w_k\}$ is any orthonormal subset of H. It is not known whether the condition (5.2) in the context of an arbitrary uniform algebra guarantees that the sequence is interpolating. (See [3].)

Proof of Theorem 5.1. We could estimate the product appearing in (5.2) directly. A more circuitous route is as follows. First observe that

$$\rho_{\mathbb{D}}(\|z\|, \|w\|) \leq \rho(z, w), \qquad z, w \in B_H, \tag{5.3}$$

where the ρ on the right is the pseudohyperbolic metric of B_H. This follows immediately from the explicit formula (3.2) for the pseudohyperbolic metric on B_H. It can also be seen directly by applying the contraction property of the pseudohyperbolic metric to a linear functional L on H satisfying $\|L\| = 1$ and $L(z) = \|z\|$, which yields $\rho_{\mathbb{D}}(\|z\|, \|w\|) \leq \rho_{\mathbb{D}}(L(z), L(w)) \leq \rho(z, w)$. By the Hayman-Newman theorem (see [13]), the condition (5.1) in $\mathbb{D}$ implies that the sequence $\{\|z_k\|\}$ is an interpolating sequence for $H^\infty(\mathbb{D})$. Hence the condition (5.2) holds for the sequence $\{\|z_k\|\}$ and the pseudohyperbolic metric $\rho_{\mathbb{D}}$, and by (5.3) this implies that the estimate (5.2) holds for the z_k's in B_H. By Theorem 5.2, the sequence $\{z_k\}$ is interpolating for $H^\infty(B_H)$, and further the interpolation constant depends only on b and the norm of the first term of the sequence. $\square$

6. A Julia-Type Estimate for Hilbert Space

Let φ be an analytic self-map of the unit ball B_E of a Banach space B such that $\varphi(0) = 0$ and $||\varphi'(0)|| < 1$. Consider the analytic function $h(\lambda)$ defined on the open unit disk $\mathbb{D}$ by $h(\lambda) = L(\varphi(\lambda w))/\lambda$, where $w \in E$ and $L \in E^*$ satisfy $\|w\| = ||L|| = 1$. Each such h satisfies $|h| \leq 1$ and $|h(0)| \leq \|\varphi'(0)||$. A normal families argument shows that for each $s < 1$, there is $c < 1$ such that any such h satisfies $|h(\lambda)| \leq c$ for $|\lambda| \leq s$. Taking the supremum over L and setting $z = \lambda w$, we obtain

$$\|\varphi(z)\| \leq c\|z\|, \qquad z \in E,\ \|z\| \leq s. \tag{6.1}$$

Hence

$$\frac{1 - \|\varphi(z)\|}{1 - \|z\|} \geq \frac{1 - c\|z\|}{1 - \|z\|}, \qquad z \in B_E,\ 0 < \|z\| < s, \tag{6.2}$$

and in fact given $0 < r < s < 1$, there is $\epsilon > 0$ such that

$$\frac{1 - \|\varphi(z)\|}{1 - \|z\|} \geq 1 + \epsilon, \qquad z \in B_E,\ r < \|z\| < s.$$

In the case of the open unit disc $\mathbb{D}$, or the open unit ball in $\mathbb{C}^n$, this estimate remains valid even as $||z|| \to 1$. We will need a variant of this Julia-type estimate for the open unit ball of Hilbert space. For the case of infinite dimensional Hilbert space, we impose a condition that restricts how z is permitted to approach the unit sphere ∂B_H of H.

We say that a subset W of B_H *approaches ∂B_H compactly* if any sequence $\{z_n\}$ in W such that $||z_n|| \to 1$ has a convergent subsequence. This occurs if and only if there is a compact subset X of ∂B_H such that if U is any open subset of H containing X, there is $\delta > 0$ such that U contains all points of W of norm $> 1 - \delta$. In particular, if W is precompact in H, then W approaches ∂B_H compactly.

Theorem 6.1. *Let H be a Hilbert space, and let φ be an analytic self-map of B_H such that $\varphi(0) = 0$ and $\|\varphi'(0)\| < 1$. Let $\delta > 0$. Suppose that W is a subset of B_H such that $\|z\| \geq \delta$ for $z \in W$, and such that W approaches*

∂B_H compactly. Then there is $\varepsilon > 0$ such that

$$\frac{1 - \|\varphi(z)\|}{1 - \|z\|} \geq 1 + \varepsilon, \qquad z \in W. \tag{6.3}$$

Proof. Suppose that the estimate (6.3) fails. Choose $\zeta_n \in W$ such that

$$\frac{1 - \|\varphi(\zeta_n)\|}{1 - \|\zeta_n\|} \to 1. \tag{6.4}$$

From (6.2) we see that $\|\zeta_n\| \to 1$ and $\|\varphi(\zeta_n)\| \to 1$. In view of the compactness hypothesis, we can pass to a subsequence and assume that $\zeta_n \to \zeta$, where $\|\zeta\| = 1$. Using the explicit formula for the pseudohyperbolic metric on B_H given in (3.2), we see that the contraction property $\rho(\varphi(z), \varphi(w)) \leq \rho(z, w)$ is equivalent to

$$\frac{|1 - \langle\varphi(z), \varphi(w)\rangle|^2}{(1 - ||\varphi(z)||^2)(1 - ||\varphi(w)||^2)} \leq \frac{|1 - \langle z, w\rangle|^2}{(1 - ||z||^2)(1 - ||w||^2)}, \qquad z, w \in B_H.$$

Substituting $w = \zeta_n$ and reorganizing terms, we obtain

$$\frac{|1 - \langle\varphi(z), \varphi(\zeta_n)\rangle|^2}{1 - ||\varphi(z)||^2} \leq \frac{1 - ||\varphi(\zeta_n)||^2}{1 - ||\zeta_n||^2} \, \frac{|1 - \langle z, \zeta_n\rangle|^2}{1 - ||z||^2}. \tag{6.5}$$

Since $|\langle\varphi(z), \varphi(\zeta_n)\rangle| \leq \|\varphi(z)\|$, we have $1 - \|\varphi(z)\| \leq |1 - \langle\varphi(z), \varphi(\zeta_n)\rangle|$, and

$$\frac{1 - \|\varphi(z)\|}{1 + ||\varphi(z)||} = \frac{(1 - \|\varphi(z)\|)^2}{1 - ||\varphi(z)||^2} \leq \frac{|1 - \langle\varphi(z), \varphi(\zeta_n)\rangle|^2}{1 - ||\varphi(z)||^2}. \tag{6.6}$$

Combining (6.5) and (6.6), and using (6.4) in passing to the limit, we obtain

$$\frac{1 - \|\varphi(z)\|}{1 + ||\varphi(z)||} \leq \frac{|1 - \langle z, \zeta\rangle|^2}{1 - ||z||^2}, \qquad z \in B_H.$$

For $z = r\zeta$, $r < 1$, this becomes

$$\frac{1 - \|\varphi(r\zeta)\|}{1 + ||\varphi(r\zeta)||} \leq \frac{(1 - r)^2}{1 - r^2} = \frac{1 - r}{1 + r}.$$

Since $(1 - r)/(1 + r)$ is a decreasing function of r for $0 < r < 1$, we obtain $\|\varphi(r\zeta)\| \geq r = \|r\zeta\|$. This contradicts (6.1), thereby establishing (6.3). $\square$

Assume as before that the analytic self-map φ of B_H satisfies $\varphi(0) = 0$ and $||\varphi'(0)|| < 1$. A finite or infinite sequence $\{z_k\}_{k\geq 0}$ is an *iteration sequence* if $\varphi(z_k) = z_{k+1}$ for $k \geq 0$. From (6.1) we see that

$$||z_0|| \geq ||z_1|| \geq ||z_2|| \geq \cdots,$$

and by induction that $||z_k|| \leq c^k ||z_0||$ for $k \geq 1$ and for some $c < 1$. Thus an infinite iteration sequence converges to 0.

Theorem 6.2. *Let H be a Hilbert space, and let φ be an analytic self-map of B_H such that $\varphi(0) = 0$ and $\|\varphi'(0)\| < 1$. Suppose that $\varphi(B_H)$ is a relatively compact subset of H. Let $\delta > 0$. Then there is $M \geq 1$ such that any finite iteration sequence $\{z_0, z_1, \ldots, z_N\}$ satisfying $z_0 \in \varphi(B_H)$ and $||z_N|| \geq \delta$ is an* *interpolating sequence for $H^\infty(B_H)$ with interpolation constant not greater than M.*

Proof. Let $W = \varphi(B_H) \cap \{||z|| > \delta\}$, and let ε be the constant of Theorem 6.1 for δ and for W. The terms $\{z_N, z_{N-1}, \ldots, z_1, z_0\}$ of any such iteration sequence (note the reversal of order) satisfy (5.1), with $b = 1/(1+\varepsilon)$ and $r = \delta$. By Theorem 5.1, the sequence is interpolating, with an interpolation constant M that depends only on φ and on δ. □

7. A Zheng Theorem for Hilbert Space

We wish to extend the main theorem from [20] to higher dimensions, and for this we focus on a Hilbert space H of finite or infinite dimensions. In the infinite dimensional case, it is natural to make some sort of compactness assumption, and we will assume that $\varphi(B_H)$ is a relatively compact subset of H. If there is $k \geq 1$ such that $\varphi_k(B_H)$ is contained in a subball ϱB_H for some $\varrho < 1$, then C_φ^k is compact, and $r_e(C_\varphi) = 0$. The following theorem addresses the alternative case.

Theorem 7.1. *Let H be a Hilbert space. Let φ be an analytic self-map of the unit ball B_H of H satisfying $\varphi(0) = 0$ and $\|\varphi'(0)\| < 1$, such that $\varphi(B_H)$ is a relatively compact subset of H. Suppose that for each $k \geq 1$, the closure of $\varphi_k(B_H)$ in H meets the unit sphere ∂B_H of H. Then the spectrum of C_φ coincides with the closed unit disk, that is, $\sigma(C_\varphi) = \overline{\mathbb{D}}$. Further, $r_e(C_\varphi) = 1$.*

Proof. Fix $m \geq 1$, to be chosen later. Consider the corresponding lower triangular decomposition of C_φ given by (3.5).

First observe that the functions in $\mathcal{P}_k$ that are bounded in modulus by 1 on B_H, are uniformly bounded on $2B_H$, and consequently are equicontinuous at each point of the closure of B_H. By the Ascoli theorem, the functions then form a compact set of continuous functions on any subset of B_H that is relatively compact in H. Since $\varphi(B_E)$ is relatively compact in H, the functions $f \circ \varphi$, f in the unit ball of $\mathcal{P}_k$, form a compact set of functions in $H^\infty(B_H)$. Hence the operator C_{kk} is a compact operator for any $k \geq 1$. By Corollary 2.4, it suffices then to show that $\sigma(Q_m) = \overline{\mathbb{D}}$, where Q_m is the restriction of C_φ to $\mathcal{R}_m$.

So fix λ such that $0 < |\lambda| < 1$. We must show that $\lambda \in \sigma(Q_m)$.

Fix δ, $0 < \delta < 1$. We will consider iteration sequences $\{z_k\}_{k=0}^\infty$ such that $z_0 \in \varphi(B_H)$ and $||z_0|| > \delta$. In view of (6.1), the norms of the elements of any such iteration sequence decrease to 0. We define $N = N(z_0)$ to be the largest integer such that $\|z_N\| > \delta$. The hypothesis guarantees that for all $k \geq 1$, $\varphi_k(B_H)$ is not contained in the ball $\{||z|| \leq \delta\}$. Consequently we can find z_0 for which $N(z_0)$ is arbitrarily large.

Choose $c < 1$ such that

$$\|\varphi(z)\| \leq c\|z\|, \qquad z \in H, \ \|z\| \leq \sqrt{\delta}.$$

We can assume that $c > \sqrt{\delta}$. By considering separately the cases $\|z_N\| \leq \sqrt{\delta}$ and $\|z_N\| > \sqrt{\delta}$, we see then also that $\|z_{N+1}\| \leq c\|z_N\|$. Since $\|z_{n+1}\| \leq c\|z_n\|$ for $n > N+1$, we obtain by induction that

$$\|z_{N+k}\| \leq c^k \|z_N\|, \qquad k \geq 0.$$

Suppose now that $\{z_k\}_{k=0}^{\infty}$ is an iteration sequence. We define the linear functional L on $\mathcal{R}_m$ by

$$L(f) = \sum_{k=0}^{\infty} \frac{f(z_k)}{\lambda^{k+1}}, \qquad f \in \mathcal{R}_m.$$

For $f \in \mathcal{R}_m$, we have that $|f(z)| \leq ||f||_\infty ||z||^m$ for all $z \in B_H$. Hence

$$\sum_{k=N+1}^{\infty} \frac{|f(z_k)|}{|\lambda|^{k+1}} \leq \sum_{k=N+1}^{\infty} \frac{\|f\|\,\|z_k\|^m}{|\lambda|^{k+1}} \leq \|f\| \frac{\|z_N\|^m}{|\lambda|^{N+1}} \sum_{k=1}^{\infty} \frac{c^{km}}{|\lambda|^k}.$$

Thus if we choose m so large that $c^m < |\lambda|$, the series defining L converges, and we obtain an estimate for the tail of the series,

$$\left| \sum_{k=N+1}^{\infty} \frac{f(z_k)}{\lambda^{k+1}} \right| \leq \|f\| \frac{\|z_N\|^m}{|\lambda|^{N+1}} \frac{c^m}{|\lambda| - c^m}, \qquad f \in \mathcal{R}_m. \tag{7.1}$$

Now choose an m-homogeneous polynomial P satisfying $\|P\| = 1$ and $|P(z_N)| = \|z_N\|^m$. By Theorem 6.2, there is $g \in H^\infty(B_H)$ such that $||g|| \leq M$, $g(z_k) = 0$ for $0 \leq k < N$, and $g(z_N) = 1$. Then $Pg \in \mathcal{R}_m$ satisfies $\|Pg\| \leq M$, and using the estimate in (7.1) for $f = Pg$, we obtain

$$|L(Pg)| \geq \left| \frac{(Pg)(z_N)}{\lambda^{N+1}} \right| - \left| \sum_{k=N+1}^{\infty} \frac{(Pg)(z_k)}{\lambda^{k+1}} \right| \geq \frac{||z_N||^m}{|\lambda|^{N+1}} - \frac{||z_N||^m}{|\lambda|^{N+1}} \frac{Mc^m}{|\lambda| - c^m}.$$

We choose m so that in addition to $c^m < |\lambda|$ we have

$$\frac{Mc^m}{|\lambda| - c^m} \; < \; \frac{1}{2},$$

and then

$$M\|L\| \geq |L(Pg)| \geq \frac{\|z_N\|^m}{2|\lambda|^{N+1}} \geq \frac{1}{2 \cdot 4^m |\lambda|^{N+1}}. \tag{7.2}$$

Next observe that for $f \in \mathcal{R}_m$,

$$((\lambda I - Q_m^*)L)(f) = \lambda L(f) - L(f \circ \varphi) = \lambda \sum_{k=0}^{\infty} \frac{f(z_k)}{\lambda^{k+1}} - \sum_{k=0}^{\infty} \frac{f(z_{k+1})}{\lambda^{k+1}} = f(z_0).$$

Hence

$$\|(\lambda I - Q_m^*)L\| \leq 1. \tag{7.3}$$

We can form iteration sequences for which N is arbitrarily large, hence by (7.2) for which $\|L\|$ is arbitrarily large. In view of (7.3), we see then that $\lambda I - Q_m^*$ is not bounded below. Consequently $\lambda I - Q_m^*$ is not invertible, and neither then is $\lambda I - Q_m$, so that $\lambda \in \sigma(Q_m)$. □

We do not know whether $\sigma_e(C_\varphi)$ is the full closed unit disk, even in the case treated by Zheng, where H is one-dimensional.

Example 7.2. There is an abundance of analytic maps satisfying the hypotheses of Theorem 7.1. Here is one family of such maps, defined on the infinite dimensional Hilbert space ℓ_2. For each $j \geq 1$ and $k \geq 2$, define a self-map φ of the unit ball of ℓ_2 by

$$\varphi(x) = (x_1^k, \ldots, x_j^k, x_{j+1}^k, \frac{x_{j+2}^k}{2}, \frac{x_{j+3}^k}{3}, \frac{x_{j+4}^k}{4}, \ldots), \qquad x \in B_{\ell_2}.$$

Evidently $\varphi(0) = 0$, $\varphi'(0) = 0$, and $\varphi(B_{\ell_2})$ is relatively compact in ℓ_2. Since $\|\varphi_n\| = 1$ for $n \geq 1$, the iterates φ_n do not converge uniformly to 0, and Theorem 7.1 shows that $\sigma(C_\varphi) = \overline{\mathbb{D}}$.

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